Table 6
Minimum sample ES variance, linear update (subscript k omitted)

Dynamical state update:

$$\hat{\psi}(+) = \hat{\psi}(-) + \mathbf{K}^{p}(\mathbf{d} - \mathbf{C}\hat{\psi}(-)). \tag{A16}$$

Sample ES Optimal Gain:

$$\mathbf{K}^{p} = \mathbf{E}_{-}\, \Pi(-) C^{p^T}(\mathbf{C}^{p}\Pi(-)\mathbf{C}^{p^T} + \mathbf{R})^{-1}, \quad \text{where } \mathbf{C}^{p} \doteq \mathbf{C}\mathbf{E}_{-}. \tag{A17}$$

Sample ES Cov. Update:

$$\mathbf{H}\,\Pi(+)\mathbf{H}^{T} = \Pi(-) - \Pi(-)\mathbf{C}^{p^T}(\mathbf{C}^{p}\Pi(-)\mathbf{C}^{p^T} + \mathbf{R})^{-1}\mathbf{C}^{p}\Pi(-). \tag{A18}$$

$$\mathbf{E}_{+} = \mathbf{E}_{-}\,\mathbf{H}. \tag{A19}$$

matrix $[E_{+}\Sigma(+), \hat{n}(+)] \in \mathbb{R}^{n\times(p+1)}$. At each assimilation time t_k, the size of the current ES is increased by one and the dominant error decomposition re-evaluated (Eqs. (A22) and (A23)).

In Eqs. (A22) and (A23), $\mathbf{E}_{+}^{\mathrm{a}}$ and $\Pi^{\mathrm{a}}(+)$ are the adapted error vectors and values, which have learned the significant tracer a posteriori residuals. In Section A.4, we refer to this adapted ES simply as the ‘‘a posteriori ES’’, without the superscript a.

A.4. State and ES forecasts

The quantities $\hat{\psi}_k(+)$, $\mathbf{E}_{k}(+)$ and $\Pi_{k}(+)$, obtained in Sections A.2 and A.3, are now forecast to t_{k+1}. The state forecast $\hat{\psi}_{k+1}(-)$ is here set to the central forecast (Eq. (A24)), which is the first-order estimate of the statistical mean state (Jazwinski, 1970). The ES is evolved by integrating to t_{k+1} an ensemble of $j = 1, \ldots, q$ perturbed states,

Table 7
Adaptive learning of the error subspace (subscript k omitted)

$$\hat{n}(+) = \mathbf{K}_{\mathrm{trc}}(\mathbf{d} - \mathbf{C}\hat{\psi}(+)), \tag{A20}$$

$$\mathbf{K}_{\mathrm{trc}} = \mathbf{E}_{\mathrm{trc}}(-)\Pi_{\mathrm{trc}}(-)\mathbf{C}_{\mathrm{trc}}^{T}(\mathbf{C}_{\mathrm{trc}}\Pi_{\mathrm{trc}}(-)\mathbf{C}_{\mathrm{trc}}^{T} + \mathbf{R})^{-1}, \quad \text{where } \mathbf{C}_{\mathrm{trc}} \doteq \mathbf{C}\mathbf{E}_{\mathrm{trc}}(-). \tag{A21}$$

$$\mathbf{E}_{+}^{a}\Sigma^{a}(+)\mathbf{V}_{+}^{aT} = \mathrm{SVD}_{p+1}([\mathbf{E}_{+}\Sigma(+), \hat{\mathbf{n}}(+)]), \tag{A22}$$

$$\Pi^{a}(+) = \frac{1}{q+1}\Sigma^{a2}(+). \tag{A23}$$