

Table 5
Nonlinear dynamical state and ES ensemble forecast

State central forecast

$$\hat{\psi}_{k+1}(-) | d\hat{\psi} = \mathbf{f}(\hat{\psi}, t) dt, \text{ with } \hat{\psi}_k = \hat{\psi}_k(+). \tag{A24}$$

ES initial conditions

$$\hat{\psi}_k^j(+) = \hat{\psi}_k(+) + E_k(+) \Pi_k^2(+) u^j, \quad j = 1, \dots, q, \tag{A25}$$

where $\mathbf{u} \in \mathbb{R}^{p \times 1}$ is random, of zero-mean and identity variance, constrained to yield $\hat{\psi}_k^j(+)$'s in accord with the measurement model (A1).

Ensemble forecast

$$d\hat{\psi}^j = \mathbf{f}(\hat{\psi}^j, t) dt, \text{ with } \hat{\psi}_k^j = \hat{\psi}_k^j(+), \quad j = 1, \dots, q. \tag{A26}$$

ES forecast

$$M_{k+1}(-) = [\hat{\psi}_{k+1}^j(-) - \hat{\psi}_{k+1}(-)], \quad j = 1, \dots, q,$$

decomposed into, $\Pi_{k+1}(-) \doteq \frac{1}{q} \Sigma_{k+1}^2(-)$ and $\mathbf{E}_{k+1}(-)$ of rank $p \leq q$, defined by,

$$\{\Sigma_{k+1}(-), \mathbf{E}_{k+1}(-) | \text{SVD}_p(\mathbf{M}_{k+1}(-)) = \mathbf{E}_{k+1}(-) \Sigma_{k+1}(-) \mathbf{V}_{k+1}^T(-)\}, \tag{A27}$$

Convergence criterion

$$\rho = \frac{\sum_{i=1}^{l_e} \sigma_i \left(\Pi^2 E^T \tilde{E} \tilde{\Pi}^2 \right)}{\sum_{i=1}^{\tilde{p}} \sigma_i(\tilde{\Pi})} \geq \alpha, \tag{A28}$$

where α is a chosen convergence limit ($1 - \epsilon \leq \alpha \leq 1$), $k = \min(\tilde{p}, p)$ and $\sigma_i(\cdot)$ selects the singular value number i .

fields, $\hat{n}(-)$, is null. The corresponding tracer error covariance decomposition, $\Pi_{\text{trc}}(-)$ and $\mathbf{E}_{\text{trc}}(-)$, is constructed following the first-stage (Eqs. (A5), (A6), (A7), (A8), (A9), (A10) and (A11)) of Section A.1. The assumptions are: (1) separability in the vertical/horizontal, (2) analytical horizontal correlation, in accord with the horizontal scales seen in the residuals, (3) vertical covariance matrix computed from the vertical EOFs of the residuals. The resulting fields $\hat{n}(+)$ are then used to update the a posteriori ES obtained in Eqs. (A18) and (A19). They form a new column of the error sample