We shall show that there exists a measurable selection for Γ which will turn out to be a random equilibrium for the random game $\mathcal{E} = \{(X_i, P_i) : i \in I\}$.

In order to apply the Aumann Measurable Selection Theorem 2.3, we need to show that Γ has a measurable graph and is nonempty valued. Since F is lower measurable, the set

$$K = \{(\omega, x) \in \Omega \times X : F(\omega, x) \neq \varnothing\} = \{(\omega, x) \in \Omega \times X : F(\omega, x) \cap X \neq \varnothing\}$$

belongs to $\Sigma \otimes \beta(X)$ and so does its complement K^c. Now observe that

$$\begin{aligned} G_\Gamma &= \{(\omega, x) \in \Omega \times X : x \in \Gamma(\omega)\} \\ &= \{(\omega, x) \in \Omega \times X : F(\omega, x) = \varnothing\} \\ &= \{(\omega, x) \in \Omega \times X : F(\omega, x) \neq \varnothing\}^c \\ &= K^c, \end{aligned}$$

and the latter set belongs to $\Sigma \otimes \beta(X)$ as it was noted above. Therefore, Γ has a measurable graph. Moreover, an appeal to [32, Theorem 6.1, p. 242] (where in [32] for each $i \in I$ and for each $x \in X$ we let $A_i(x) = \overline{A}_i(x) = X_i$) shows that for each $\omega \in \Omega$ we have $\Gamma(\omega) \neq \varnothing$. Therefore, by the Aumann Measurable Selection Theorem there exists a measurable function $x^*: \Omega \to X$ such that $x^*(\omega) \in \Gamma(\omega)$ for almost all $\omega \in \Omega$, i.e., $F(\omega, x^*(\omega)) = \varnothing$ for almost all $\omega \in \Omega$. The latter implies that for each $i \in I$ we have $P_i(\omega, x^*(\omega)) = \varnothing$ for almost all $\omega \in \Omega$, i.e., x^* is a random equilibrium for the game $\mathcal{E}$. ∎

Note that in Theorem 7.1 the assumption that (Ω, Σ, μ) is a complete finite measure space can be replaced by the fact that (Ω, Σ) is a measurable space. The proof remains the same. In particular, since for each fixed $\omega \in \Omega$ the correspondence $F(\omega, \cdot): X \to 2^X$ has open lower sections, it is also l.s.c. ([32, Proposition 4.1, p. 237]) and therefore Γ is closed valued. Since Γ has a measurable graph and is closed valued, it is also lower measurable [15, Theorem 3.3, p. 60]. One can now appeal to the Kuratowski and Ryll–Nardzewski Measurable Selection Theorem to complete the proof of Theorem 7.1.

Finally, note that assumption (4′) of Theorem 7.1 is weaker than assumption (4) of Theorem 3.4 and assumption (2′) is different from assumption (2) of the same theorem. Hence, neither result implies the other. However, the methods of proof are different. It can be easily seen that Corollary 3.5 follows directly from Theorem 7.1. The idea of the proof is identical with the one used to prove Corollary 3.3.

Remarks. (A) The form of the Bayesian game defined in Section 4 can be generalized by replacing each player's random payoff function $h_i: \Omega \times X \to R$ by a random preference correspondence $P_i: \Omega \times X \to 2^{X_i}$. Following the notation of Section 4, in this new setting the conditional expected payoff $F_i(\omega, x)$ of each player is the integral of the correspondence P_i, i.e.,

$$F_i(\omega, x) = \int_{E(\omega)} q_i(t | E(\omega)) P_i(t, x)\, d\mu(t).$$