For each i define the correspondence $\varphi_i: \int \hat{X}_i^e \to 2^{\int X_i^e}$ by

$$\varphi(\hat{x}_i) = \left\{ y_i \in \int X_i^e : v_i(\omega, y_i, \hat{x}_i) = \max_{x_i \in \int X_i^e} v_i(\omega, x_i, \hat{x}_i) \text{ for almost all } \omega \in \Omega \right\}.$$

Also, define $F: \int X^e \to 2^{\int X^e}$ by $F(x) = \Pi_{i=1}^n \varphi_i(\hat{x}_i)$. Note that by repeating the proof of Theorem 5.1, we can see that each φ_i is nonempty, closed, and convex valued and u.s.c. Clearly, a fixed point of F is a pure strategy Bayesian equilibrium for the game $\mathcal{G}$. If we establish that $\int X = \int X^e$ and that $\int X$ is compact, convex and nonempty, then we are done.

Since X_i is a compact convex set, it follows from the Krein–Milman Theorem that $\text{co}(X_i^e) = X_i$. By [4, Theorem 3, p. 2], we have $\int \text{co}(X_i^e) = \int X_i^e = \int X_i$, and therefore $\int X = \int X^e$. Moreover, by [4, Theorem 4, p. 2], $\int X_i$ is compact. Hence $\int X_i$ is compact, convex and nonempty (the nonemptiness follows from the Measurable Selection Theorem), and so is $\int X$. Now by the Kakutani Fixed Point Theorem there exists $x^* \in \int X^e$ such that $x^* \in F(x^*)$, i.e., x^* is a pure strategy equilibrium for the game $\mathcal{G}$. ∎

If assumption (2) of Theorem 5.2 is replaced by

(2′) $X_i: \Omega \to 2^Y$ (where Y is a separable Banach space whose dual has the RNP) is a nonempty, weakly compact, convex and integrably bounded correspondence having a measurable graph,

then only an approximate pure strategy Bayesian equilibrium can be obtained. The reason is that Aumann's Theorem 3 in [4] is no longer true. (See, for instance, [29] or [36] for a counterexample.) In particular, in this case we have only that $\int X = \int \overline{\text{co}}(X^e) = \overline{\int X^e}$. Moreover, by [35, Lemma 31, p. 307], $\int X$ is weakly compact. Carrying out now the argument outlined in the proof of Theorem 5.2 one can easily prove the existence of an approximate pure strategy Bayesian equilibrium. For other results on approximate purification of mixed strategies see [5, 20, 27, 28].

We close the section by mentioning that all the equilibrium existence results in Sections 3 and 4 can be easily extended to abstract economies as defined in [10], [31], and [32]. Moreover, one can use the equilibrium results for abstract economies to obtain equilibrium existence theorems for random exchange economies or Bayesian exchange economies. In particular, in this setting of incomplete information the appropriate equilibrium notion is that of a rational expectations equilibrium. We hope to take up these details, however, in a subsequent paper.

6. Related literature

The equilibrium existence results for games with incomplete information which are related to Theorems 4.2 and 5.1 that we know of, are those in E. J. Balder [6],