5. *each h_i is integrably bounded.*

Then the game $\mathcal{G}$ has a Bayesian equilibrium.

Proof: The result follows directly from Corollary 3.5. To see this, note that since each $h_i(\omega,\cdot)$ is continuous and h_i is integrably bounded by virtue of the Lebesgue Dominated Convergence Theorem, we can automatically conclude that the function

$$v_i(\omega,\cdot) = \int_{E(\omega)} q_i\big(t|E(\omega)\big)h(t,\cdot)\,d\mu(t)$$

is continuous, where

$$q_i\big(t|E(\omega)\big) = \begin{cases} 0, & \text{if } t \notin E(\omega); \\ \frac{q_i(t)}{\int_{E(\omega)} q_i(s)\,d\mu(s)}, & \text{if } t \in E(\omega)\,. \end{cases}$$

Furthermore, it can be easily seen that each function $v_i(\cdot,x)$ is S-measurable. Finally, it follows from (4) that for each $\omega \in \Omega$ and each $\hat{x}_i \in \hat{X}_i$ that the function $v_i(\omega,x_i,\hat{x}_i)$ is concave in x_i. We consider the Bayesian game $\mathcal{G} = \{(X_i,h_i,S,q_i) : i \in I\}$ as a random game $\mathcal{E} = \{(X_i,v_i) : i \in I\}$. Obviously, the existence of a random Nash equilibrium for the game $\mathcal{E}$ implies the existence of a Bayesian equilibrium for the game $\mathcal{G}$. It can be easily seen that the random game $\mathcal{E}$, satisfies the assumptions of Corollary 3.5 and consequently, the game $\mathcal{E}$ has a random Nash equilibrium.[4] Hence, there exists an S-measurable function $x^*:\Omega \to X$ such that

$$v_i\big(\omega,x^*(\omega)\big) = \max_{y_i\in X_i} v_i\big(\omega,x_1^*(\omega),\ldots,x_{i-1}^*(\omega),y_i,x_{i+1}^*(\omega),\ldots\big)$$

for almost all $\omega \in \Omega$ and all $i \in I$. In other words, x^* is a Bayesian equilibrium for the game $\mathcal{G} = \{(X_i,h_i,S,q_i) : i \in I\}$, and the proof of the theorem is finished. ∎

5. Asymmetric Bayesian games

We now turn our attention to the rather more interesting case where the information set of each player is different.

Let $\mathcal{G} = \{(X_i,h_i,S,q_i) : i \in I\}$ be a Bayesian game as described before. Denote by L_{X_i} the set of all Bochner integrable and S_i-measurable selections from the strategy set X_i of player i, i.e.,

$$L_{X_i} = \{x_i \in L_1(\mu,Y) : x_i \text{ is } S_i\text{-measurable and } x_i(\omega) \in X_i \text{ for } \mu\text{-a.e. } \omega\}.$$

[4] Note that the proofs of Theorems 3.2 and 3.4 and Corollaries 3.3 and 3.5 remain unchanged if the measurability assumptions on either the preference correspondence P_i or the payoff function u_i of each player are made with respect to the algebra generated by the partition S_i instead of Σ.