Proof: For each $i \in I$, define the correspondence $Q_i: \Omega \times X \to 2^{X_i}$ by

$$Q_i(\omega, x) = \{y_i \in X_i : h_i(\omega, x, y_i) > 0\},$$

where $h_i(\omega, x, y_i) = u_i(\omega, y_i, \hat{x}_i) - u_i(\omega, x)$. Letting $S = \Omega \times X$, $Z = X_i$, $\alpha = \Sigma \otimes \beta(X)$, $g(s, z) = h_i(\omega, x, y_i)$, $K(s) = Q_i(\omega, x)$ for $s = (\omega, x)$ in Lemma 2.13(b), we can conclude that each Q_i is lower measurable. It follows from assumption (4) that each Q_i is convex valued, and clearly for any measurable function $x: \Omega \to X$ we have $x_i(\omega) \notin \operatorname{co} Q_i(\omega, x(\omega)) = Q_i(\omega, x(\omega))$ for almost all $\omega \in \Omega$. Moreover, it follows from assumption (2) that each set-valued function $Q_i(\omega, \cdot)$ has an open graph in $X \times X_i$. Hence, the random game $E = \{(X_i, Q_i) : i \in I\}$ satisfies the assumptions of Theorem 3.2 and therefore E has a random equilibrium. That is, there exists a measurable function $x^*: \Omega \to X$ such that $Q_i(\omega, x^*(\omega)) = \varnothing$ for almost all $\omega \in \Omega$ and all $i \in I$. But this implies

$$u_i(\omega, x^*(\omega)) = \max_{y_i \in X_i} u_i(\omega, x_1^*(\omega), \ldots, x_{i-1}^*(\omega), y_i, x_{i+1}^*(\omega), \ldots),$$

for almost all $\omega \in \Omega$ and all $i \in I$, i.e., x^* is a random Nash equilibrium for the game $\Gamma = \{(X_i, u_i) : i \in I\}$, as claimed. ∎

We now provide an extension of Theorem 3.2 to strategy sets which may be subsets of a separable Banach space.

Theorem 3.4 *Let $\mathcal{E} = \{(X_i, P_i) : i \in I\}$ be a random game satisfying for each i the following assumptions:*

1. *each X_i is a nonempty, compact, and convex subset of a separable Banach space,*

2. *each $\operatorname{co} P_i$ has a measurable graph, i.e.,*

$$\{(\omega, x, y_i) \in \Omega \times X \times X_i : y_i \in \operatorname{co} P_i(\omega, x)\} \in \Sigma \otimes \beta(X) \otimes \beta(X_i),$$

3. *for every measurable function $x: \Omega \to X$ we have $x_i(\omega) \notin \operatorname{co} P_i(\omega, x(\omega))$ for almost all $\omega \in \Omega$, and*

4. *for each $\omega \in \Omega$ the set-valued function $P_i(\omega, \cdot)$ has an open graph in $X \times X_i$.*

Then $\mathcal{E}$ has a random equilibrium.

Proof: For each $i \in I$ define the correspondence $\varphi_i: \Omega \times X \to 2^{X_i}$ by $\varphi_i(\omega, x) = \operatorname{co} P_i(\omega, x)$. Since, by assumption (4), each $P_i(\omega, \cdot)$ has an open graph in $X \times X_i$, it can be easily checked (see [33, Lemma 4.1]) that so does $\varphi_i(\omega, \cdot)$ for each $\omega \in \Omega$.

Let $O_i = \{(\omega, x) \in \Omega \times X : \varphi_i(\omega, x) \neq \varnothing\}$. Since φ_i has a measurable graph (by hypothesis (2)) and it is convex valued, Theorem 2.5 guarantees the existence of a Carathéodory selection φ_i. To complete the proof now proceed as in the proof of Theorem 3.2. ∎