- $f_i(\omega, x) \in \varphi_i(\omega, x)$ for all $(\omega, x) \in O_i$,
- for each $x \in X$ the function $f_i(\cdot, x)$ is measurable on O_i^x, and
- for each $\omega \in \Omega$ the function $f_i(\omega, \cdot)$ is continuous on O_i^ω.

Now for each $i \in I$ define the correspondence $F_i: \Omega \times X \to 2^{X_i}$ by

$$F_i(\omega, x) = \begin{cases} \{f_i(\omega, x)\}, & \text{if } (\omega, x) \in O_i; \\ X_i, & \text{if } (\omega, x) \notin O_i . \end{cases}$$

Clearly, F is nonempty, closed, and convex valued and by Lemma 2.12 it is also lower measurable. Since for each fixed $\omega \in \Omega$ the function $\varphi_i(\omega, \cdot)$ is l.s.c., the set

$$O_i^\omega = \{x \in X : \varphi_i(\omega, x) \neq \varnothing\} = \{x \in X : \varphi_i(\omega, x) \cap X_i \neq \varnothing\}$$

is open in the relative topology of X, and consequently for each fixed $\omega \in \Omega$ the function $F_i(\omega, \cdot)$ is u.s.c.; see [32, Lemma 6.1].

Next, define the correspondence $F: \Omega \times X \to 2^X$ by $F(\omega, x) = \Pi_{i \in I} F_i(\omega, x)$. Clearly, F is nonempty, closed and convex valued. Since each F_i is lower measurable, it follows from Theorem 2.6 that F is lower measurable as well. By Theorem 2.2, the correspondence $F(\omega, \cdot): X \to 2^X$ is u.s.c. for each $\omega \in \Omega$. Furthermore, F satisfies the hypotheses of Corollary 2.11 and consequently there exists a random fixed point, i.e., there exists a measurable function $x^*: \Omega \to X$ such that $x^*(\omega) \in F(\omega, x^*(\omega))$ for almost all $\omega \in \Omega$.

Finally, we shall show that the random fixed point is by construction a random equilibrium for the game E. Notice that if $(\omega, x^*(\omega)) \in O_i$ for all $\omega \in S$ with $\mu(S) > 0$, then by the definition of F_i, we have $x_i^*(\omega) = f_i(\omega, x^*(\omega)) \in \operatorname{co} P_i(\omega, x^*(\omega))$, contrary to assumption (3). Thus, $(\omega, x^*(\omega)) \notin O_i$ holds for almost all $\omega \in \Omega$ and all $i \in I$. In other words, we have $\operatorname{co} P_i(\omega, x^*(\omega)) = \varnothing$ for almost all $\omega \in \Omega$ and all i, which in turn implies that $P_i(\omega, x^*(\omega)) = \varnothing$ for almost all $\omega \in \Omega$ and all $i \in I$. That is, $x^*: \Omega \to X$ is a random equilibrium for E, and the proof of the theorem is finished. ∎

As a corollary of Theorem 3.2 we obtain a generalized random version of Nash's equilibrium existence result [22, Theorem 1, p. 288].

Corollary 3.3 *Let $\Gamma = \{(X_i, u_i) : i \in I\}$ be a Nash-type random game satisfying for each i the following assumptions:*

1. *each X_i is a nonempty, compact, and convex subset of R^ℓ,*
2. *for each fixed $\omega \in \Omega$ the function $u_i(\omega, \cdot)$ is continuous,*
3. *for each fixed $x \in X$ the function $u_i(\cdot, x)$ is measurable, and*
4. *for each $\omega \in \Omega$ and each $\hat{x}_i \in \hat{X}_i$ the function $u_i(\omega, x_i, \hat{x}_i)$ is quasiconcave in x_i.*

Then there exists a random Nash equilibrium for Γ.