Proof: (a) Since for each fixed $s \in S$ the function $g(s,\cdot)$ is continuous and for each fixed $z \in Z$ the function $g(\cdot,z)$ is measurable, it follows from a standard result that g is jointly measurable. Observe that

$$\begin{aligned} g^{-1}((0,\infty)) &= \{(s,z) \in S \times Z : g(s,z) > 0\} \\ &= \{(s,z) \in S \times Z : z \in K(s)\} \\ &= G_K, \end{aligned}$$

and the latter set belongs to $\alpha \otimes \beta(Z)$ since g is jointly continuous. (b) We must show that the set $\{s \in S : K(s) \cap V \neq \varnothing\}$ belongs to α for every open subset V of Z. As it was remarked above, g is jointly measurable, i.e., g is measurable with respect to the product σ-algebra $\alpha \otimes \beta(Z)$. Let D be a countable dense subset of Z, and let $U = (0,\infty)$. Observe that

$$\begin{aligned} \{s \in S : K(s) \cap V \neq \varnothing\} &= \{s \in S : g(s,z) \in U \text{ for some } z \in V\} \\ &= \{s \in S : g(s,d) \in U \text{ for some } d \in D\} \\ &= \bigcup_{d \in D} \{s \in S : g(s,d) \in U\}, \end{aligned}$$

and the latter set belongs to α since for each fixed $z \in Z$ the function $g(\cdot,z)$ is measurable. This completes the proof of the lemma. ∎

The notions that will be introduced next are quite standard (see for instance N. C. Yannelis [36]) but we briefly outline them for the sake of completeness.

We begin by defining the notion of a Bochner integrable function. Let (T,Σ,μ) be a finite measure space and Y be a Banach space. A function $f: T \to Y$ is called *simple* if there exist $y_1, y_2, \ldots, y_n$ in Y and $A_1, A_2, \ldots, A_n$ in Σ such that $f = \sum_{i=1}^{n} y_i \chi_{A_i}$, where χ_{A_i} denotes the characteristic function of the set A_i. A function $f: T \to Y$ is said to be *μ-measurable* if there exists a sequence of simple functions $\{f_n\}$ such that $\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0$ for almost all $t \in T$. A μ-measurable function $f: T \to Y$ is said to be *Bochner integrable* if there exists a sequence of simple functions $\{f_n\}$ such that

$$\lim_{n\to\infty} \int_T \|f_n(t) - f(t)\| \, d\mu(t) = 0.$$

In this case, the integral of f over a set $E \in \Sigma$ is defined by

$$\int_E f(t)\, d\mu(t) = \lim_{n\to\infty} \int_E f_n(t)\, d\mu(t).$$

It can be shown that if $f: T \to Y$ is a μ-measurable function, then f is Bochner integrable if and only if $\int_T \|f(t)\|\, d\mu(t) < \infty$. We denote by $L_1(\mu, Y)$ the space of equivalence classes of Y-valued Bochner integrable functions $x: T \to Y$ normed by $\|x\| = \int_T \|x(t)\|\, d\mu(t)$. It can be easily shown that $L_1(\mu, Y)$ under the norm $\|\cdot\|$ is a Banach space.

A Banach space Y has the *Radon–Nikodym Property with respect to the measure space* (T,Σ,μ) if for each μ-continuous vector measure $G: \Sigma \to Y$ of bounded