the correspondence $\varphi: \Omega \to 2^X$ is closed-valued. Since φ is closed-valued and it has a measurable graph by [15, Theorem 3.3, p. 56], φ is lower measurable. One can now appeal to the Kuratowski and Ryll–Nardzewski Measurable Selection Theorem [15, p. 60] to complete the proof of Theorem 2.10.

We continue with two more results that will be needed later.

Lemma 2.12 *Let (S, α, μ) be a complete measure space, let X and Y be separable metric spaces, and let $\varphi: S \times X \to 2^Y$ be a lower measurable (possibly empty valued) correspondence. Suppose that for each fixed $s \in S$ the function $\varphi(s, \cdot)$ is l.s.c. Put $O = \{(s,x) \in S \times X : \varphi(s,x) \neq \emptyset\}$ and let $f: O \to Y$ be a Carathéodory selection for φ. Then the correspondence $\theta: S \times X \to 2^Y$, defined by*

$$\theta(s,x) = \begin{cases} \{f(s,x)\}, & \text{if } (s,x) \in O_i; \\ Y, & \text{if } (s,x) \notin O_i \;, \end{cases}$$

is lower measurable.

Proof: We begin by making a couple of observations. First notice that since φ is lower measurable, the set $O = \{(s,x) \in S \times X : \varphi(s,x) \cap Y \neq \emptyset\}$ belongs to $\alpha \otimes \beta(X)$. By Theorem 2.9 for each $x \in X$ the set

$$\begin{aligned} O^x &= \{s \in S : (x,s) \in O\} \\ &= \text{proj}_S\left[\{(s,x) \in S \times X : \varphi(s,x) \neq \emptyset\} \cap (S \times \{x\})\right] \\ &= \text{proj}_S\left[O \cap (S \times \{x\})\right], \end{aligned}$$

belongs to α. Moreover, note that since for each fixed $s \in S$ the function $\varphi(s, \cdot)$ is l.s.c., it follows that for each $s \in S$ the set $O^s = \{x \in X : (s,x) \in O\}$ is open in X. Since for each fixed $s \in S$ the function $f(s, \cdot)$ is continuous on O^s and for each $x \in X$ the function $f(\cdot, x)$ is measurable on O^x, by Theorem 2.8 the function f is jointly measurable. Now it can be easily seen that for every open subset V of Y the set $A = \{(s,x) \in S \times X : \theta(s,x) \cap V \neq \emptyset\} = B \cup C$, where $B = \{(s,x) \in O : f(s,x) \in V\}$ and $C = \{(s,x) \in S \times X \setminus O : Y \cap V \neq \emptyset\}$. Clearly, $B \in \alpha \otimes \beta(X)$ and $C \in \alpha \otimes \beta(X)$ and therefore $A = B \cup C$ belongs to $\alpha \otimes \beta(X)$. Consequently, θ is lower measurable, as claimed. ∎

Lemma 2.13 *Let (S, α) be a measurable space, Z be a separable metric space and R^* be the extended real line. Let $g: S \times Z \to R^*$ be a function such that for each fixed $s \in S$ the function $g(s, \cdot)$ is continuous and for each fixed $z \in Z$ the function $g(\cdot, z)$ is measurable. If $K: S \to 2^Z$ is the correspondence defined by*

$$K(s) = \{z \in Z : g(s,z) > 0\},$$

then we have:

a. *$G_K \in \alpha \otimes \beta(Z)$, i.e., K has a measurable graph, and*

b. *K is lower measurable.*