We now proceed to verify the quasi-equilibrium conditions. The argument is similar to Bewley's (1974), but more complicated, since the endowments e_i^γ may differ from the endowments e_i. First of all, we know that $\sum x_i^\gamma = \sum e_i^\gamma$ for each γ. By construction. the endowments e_i^γ converge to e_i in the norm topology and hence in the topology τ (which is weaker than the norm topology). Since the vectors x_i^γ converge to $\bar{x}_i$ in the topology τ, and τ is a vector space topology we conclude that $\sum \bar{x}_i = \sum e_i$.

Now let us suppose that $y_i \in P_i(\bar{x}_1,\dots,\bar{x}_N)$ snd that $\bar{\pi}(y_i) < \bar{\pi}(e_i)$. Proceeding exactly as in the construction of extremely desirable commodities, we find vectors $y_i^\gamma \in F_\gamma^+$ such that $\|y_i^\gamma - y_i\|$ tends to 0 (with γ). Since the preference relation P_i is (τ, norm)-continuous, we conclude that $y_i^\gamma \in P_i(x_1^\gamma,\dots,x_N^\gamma)$ if γ is large enough. Since the vectors in question all belong to F_γ, it follows that $y_i^\gamma \in P_i^\gamma(x_1^\gamma,\dots,x_N^\gamma)$. On the other hand, since $\|y_i^\gamma - y_i\|$ and $\|e_i^\gamma - e_i\|$ both tend to 0 (with γ), and $\{\pi^\gamma\}$ converges to $\bar{\pi}$ in the weak-star topology, we may apply Lemma A of the appendix again to conclude that $\pi^\gamma(y_i^\gamma) < \pi^\gamma(e_i^\gamma)$ for γ sufficiently large. Since $y_i^\gamma \in P_i^\gamma(x_1^\gamma,\dots,x_N^\gamma)$ for large γ, this contradicts the fact that $(x_1^\gamma,\dots,x_N^\gamma, \pi^\gamma)$ is a quasi-equilibrium for $\mathcal{E}^\gamma$. We conclude that, if $y_i \in P_i(\bar{x}_i,\dots,\bar{x}_N)$, then $\bar{\pi}(y_i) \geqq \bar{\pi}(e_i)$.

Finally, we need to show that $\bar{\pi}(\bar{x}_i) = \bar{\pi}(e_i)$ for each i. But if $\bar{\pi}(\bar{x}_i) \neq \bar{\pi}(e_i)$ for some i, we must have $\bar{\pi}(\bar{x}_j) < \bar{\pi}(e_j)$ for some j, since $\sum \bar{x}_i = \sum e_i$. On the other hand, $\bar{x}_j + tv_j \in P_j(\bar{x}_1,\dots,\bar{x}_N)$ for each $t > 0$ (since v_j is extremely desirable) and $\bar{\pi}(\bar{x}_j + tv_j) < \bar{\pi}(e_j)$ for t sufficiently small (since $\bar{\pi}(x_j) < \bar{\pi}(e_j)$). This contradicts the conclusion of the previous paragraph. This completes the proof that $(\bar{x}_1,\dots,\bar{x}_N,\bar{\pi})$ is a quasi-equilibrium.

It remains to show that, when $\mathcal{E}$ is irreducible, every quasi-equilibrium $(x_1^*,\dots,x_N^*;\pi^*)$ is actually an equilibrium. Since π^* is a non-zero price, $\pi^*(z) \neq 0$ for some positive z. Since e is strictly positive, $\|(ne \wedge z) - z\|$ is small provided n is large. Hence $\pi^*(ne \wedge z) \neq 0$, so that $\pi^*(1/n)(ne \wedge z) \neq 0$ while $(1/n)(ne \wedge z) \leqq e$; Lemma 7.1 now implies that $(x_1^*,\dots,x_N^*,\pi^*)$ is actually an equilibrium. This completes the proof. □

It is worth noting that, by the same argument, we can show that the set of all quasi-equilibria $(\bar{x}_1,\dots,\bar{x}_N;\bar{\pi})$ with $\|\pi\| \leqq 1$ is a compact subset of $L \times L \cdots \times L \times L'$, where we give L the topology τ and L' the weak-star topology. (Of course, if $\mathcal{E}$ is irreducible, the set of equilibria is compact, since it coincides with the set of quasi-equilibria.)

8. Concluding remarks

Remark 1. Throughout, we have assumed that the consumption set of each agent is the positive cone L^+. An examination of the proof will show, however, that it works equally well for (some) other consumption sets. For example, it would suffice to assume that the consumption set X_i of the ith agent has the properties: