this relation can be extended[7] to a reflexive, antisymmetric, transitive relation $\leqq$ on D (i.e., a *partial ordering*) which is given by

$\alpha \leqq \beta$ if either (i) $\alpha = \beta$, or (ii) there are elements $\gamma_1, \dots, \gamma_k$ of D such that

$$\alpha = \gamma_1, \quad \beta = \gamma_k \quad \text{and} \quad \gamma_1 < \gamma_2 \cdots < \gamma_k.$$

A simple calculation, using (ii), the triangle inequality and the fact that $\sum_{n=1}^{\infty} 2^{-n} = 1$, shows that if $\alpha \leqq \beta$ then

$$\operatorname{dist}(z, F_\beta^+) \leqq 4 \cdot s^{-n_\alpha} \|z\|$$

for each $z \in F_\alpha^+$. We next show that the partial ordering $\leqq$ actually directs the set D.

Lemma 7.3. The set D of special configurations, equipped with the partial ordering $\leqq$, is a directed set. That is, if α and β belong to D then there is a γ in D for which $\alpha \leqq \gamma$ and $\beta \leqq \gamma$.

Proof. Since F_α and F_β are finite-dimensional, their unit spheres are compact. Hence we can choose finite sets of vectors $\{x_1, \dots, x_J\} \subset F_\alpha$ and $\{y_1, \dots, y_K\} \subset F_\beta$ such that $\|x_j\| = 1$ for each j; $\|y_k\| = 1$ for each k; for each $x \in F_\alpha$ with $\|x\| = 1$, there is an index j with $\|x - x_j\| < 2^{-2-n_\alpha}$ and for each $y \in F_\beta$, with $\|y\| = 1$, there is an index k with $\|y - y_k\| < 2^{-2^-} n^\beta$. We now use Lemma 7.2 to choose a special configuration γ such that $1/n_\gamma < 1/(n_\alpha + n_\beta)$ (which means $n_\gamma > n_\alpha + n_\beta$), $\operatorname{dist}(x_j, F_\gamma^+) < 2^{-n_\alpha - 2}$ for each j and $\operatorname{dist}(y_k, F_\gamma^+) < 2^{-n_\beta - 2}$ for each k. Since the norm of F_γ is positively homogeneous, the triangle inequality and our choice of $\{x_1, \dots, x_J\}$ and $\{y_1, \dots, y_K\}$ imply that $\operatorname{dist}(x, F_\gamma^+) \leqq 2^{-n_\alpha} \|x\|$ for each $x \in F_\alpha^+$ and $\operatorname{dist}(y, F_\gamma^+) \leqq 2^{-n_\beta} \|y\|$ for each $y \in F^{+\beta}$. Thus $\alpha < \gamma$ and $\beta < \gamma$; in particular, $\alpha \leqq \gamma$ and $\beta \leqq \gamma$, as desired. □

With all of the preliminary constructions out of the way, we now turn to the main argument.

Proof of the Main Existence Theorem. For each special configuration $\alpha = (F_\alpha, n_\alpha, e_1^\alpha, \dots, e_N^\alpha)$ we define consumption sets X_i^α and preference relations $p_i^\alpha \colon \prod X_j^\alpha \to 2^{X_i^\alpha}$ by

[7]Any acyclic relation may always be extended to a reflexive, antisymmetric transitive relation by exactly this procedure.