Fix $i \in I$ and a vector ζ with $0 \leqq \zeta \leqq e_i$ and $\pi(\zeta) \neq 0$. Let $y_i \in P_i(x_1, \ldots, x_N)$; then $\pi(y_i) \geqq \pi(e_i)$, and we want to show that in fact $\pi(y_i) > \pi(e_i)$. If $\pi(y_i) = \pi(e_i)$, we distinguish three cases. *Case 1.* $\pi(e_i) > 0$. Then $\pi(ty_i) < \pi(e_i)$ for $t < 1$, while $ty_i \in P_i(x_1, \ldots, x_N)$ if t is close to 1 (by continuity of P_i). This violates the quasi-equilibrium conditions. *Case 2.* $\pi(e_i) < 0$. Then $\pi(sy_i) < \pi(e_i)$ for $s > 1$ and $sy_i \in P_i(x_1, \ldots, x_N)$ for s near 1, again violating the quasi-equilibrium conditions. *Case 3.* $\pi(e_i) = 0$. By continuity of P_i, for all small real numbers $r > 0$ we have $y_i + re_i \in P_i(x_1, \ldots, x_N)$. Since $0 \leqq \zeta \leqq e_i$, we know that $y_i + re_i + r^*\zeta \in P_i(x_1, \ldots, x_N)$ provided that $|r^*|$ is sufficiently small (if $|r^*| < r$ then $y_i + re_i + r^*\zeta \geqq 0$). On the other hand, $\pi(y_i + re_i + r^*\zeta) = r^*\zeta) = r^*\pi(\zeta)$, and $r^*\pi(\zeta) < 0 = \pi(e_i)$ if r^* and $\pi(\zeta)$ have opposite signs. This again violates the quasi-equilibrium conditions. We conclude that the equilibrium conditions hold for all agents in I.

Notice that I is not empty. For, since $0 \leqq z \leqq \sum e_i$, we may use the Riesz Decomposition Property to write $z = \sum z_i$ with $0 \leqq z_i \leqq e_i$ for each i. Then $\pi(z) = \sum \pi(z_i) \neq 0$, so $\pi(z_i) \neq 0$ for at least one agent i, and this agent belongs to I.

Finally we show that J is empty. For, if not, irreducibility of $\mathcal{E}$ guarantees that there is an $i \in I$, a $j \in J$ and a $\zeta \in L$ such that $0 \leqq \zeta \leqq e_j$ and $x_i + \zeta \in P_i(x_1, \ldots, x_N)$. Since $j \in J$ we know that $\pi(\zeta) = 0$, so that $\pi(x_i + \zeta) = \pi(x_i) \leqq \pi(e_i)$; this violates the equilibrium conditions just established for agent i. We conclude that J is empty, and hence that $(x_1, \ldots, x_N, \pi)$ is an equilibrium. This completes the proof of Lemma 7.1. □

Throughout the remainder of this section, we assume that all the hypotheses of the Main Existence Theorem are satisfied (except for irreducibility of the economy $\mathcal{E}$).

We are going to obtain a quasi-equilibrium for $\mathcal{E}$ as a limit of equilibria of subeconomies whose commodity spaces are finite-dimensional vector sublattices of L. Because the family of finite-dimensional vector sublattices L is not directed by inclusion, we need to carry along some extra information. The precise structure we need is an $(N+2)$-tuple $\alpha = (F_\alpha, n_\alpha, e_1^\alpha, \ldots, e_N^\alpha)$, where F_α is a finite-dimensional vector sublattice of L, n_α is a positive integer and $e_1^\alpha, \ldots, e_N^\alpha$ are positive elements of F_α which satisfy

(a) $\sum_{i=1}^N e_i^\alpha$ is strictly positive in F_α,
(b) $\|e_i - e_i^\alpha\| < 1/n_\alpha$ for each i,
(c) $e_i^\alpha \leqq e_i$ for each i.

We shall call such an $(N+2)$-tuple α a *special configuration.* (Notice that, since F_α is a vector sublattice of L, it follows that $\sum e_i^\alpha$ is a positive element of L, but it need not be strictly positive in L.) The vectors e_i^α will be the