$$e_i^* = \sum_{l=1}^{k} c_{il} \chi_{V_l},$$

$$b_j^* = \sum_{l=1}^{k} d_{jl} \chi_{V_l}.$$

This construction guarantees that the functions e_i^*, b_j^* are positive and satisfy the following inequalities for each w in Ω:

$$e_i^*(w) \leqq e_i(w) \quad \text{and} \quad b_j^*(w) \leqq b_j(w),$$

$$|e_i^*(w) - e_i(w)| \leqq \delta' \quad \text{and} \quad |b_j^*(w) - b_j(w)| \leqq \delta'.$$

The first two inequalities imply that $0 \leqq e_i^* \leqq e_i$ and $0 \leqq b_j^* \leqq b_j$. The second two inequalities, together with the fact that $e(w) = 1$ for each w, imply that $|e_i^* - e_i| \leqq \delta' e$ and $|b_j^* - b_j| \leqq \delta' e$. By the lattice property of the norm (and the fact that $\delta' \|e\| < \delta$) this yields

$$\|e_i^* - e_i\| < \delta \quad \text{and} \quad \|b_j^* - b_j\| < \delta.$$

It remains only to show that $\sum e_i^*$ is strictly positive in K. Equivalently, we must show that for each l, at least one of the coefficients c_{il} is strictly positive. Fix a point w_l in V_l. Since $e(w_l) = 1$, there is at least one e_i such that $e_i(w_l) \geqq 1/N$. Since the variation of e_i on V_l is at most δ', this means that $c_{il} \geqq (1/N) - \delta' > 0$, as required. This completes the proof of Theorem 6.1. □

7. Proof of the Main Existence Theorem

We begin by isolating parts of the argument as lemmas. The first one will be useful elsewhere, so we establish an appropriately general version; it is closely related to a finite-dimensional result of McKenzie (1959).

Lemma 7.1. Let $\mathcal{E} = \{(X_i, P_i, e_i): i = 1, 2, \ldots, N\}$ be an irreducible economy in the Banach lattice L. Assume that $X_i = L^+$ for each i, and that the preference relation P_i is (norm, norm) continuous for each i. If $((x_1, \ldots, x_n), \pi)$ is a quasi-equilibrium for $\mathcal{E}$ and there is a vector $z \in L$ such that $0 \leqq z \leqq \sum e_i$ and $\pi(z) \neq 0$, then $(x_1, \ldots, x_N, \pi)$ is actually an equilibrium.

Proof. Let I denote the set of agents i for which there is a vector ζ with $0 \leqq \zeta \leqq e_i$ and $\pi(\zeta) \neq 0$; let J denote the complementary set of agents. We first show that the equilibrium conditions are satisfied for all agents in I.