We stress that the Price Lemma depends crucially on the facts that L is a lattice and that the norm on L is a lattice norm, and may fail to be true if these assumptions are not satisfied. For example, the Price Lemma may fail if L is the two-dimensional vector lattice $\mathbb{R}^2$, equipped with a vector-space norm which is not a lattice norm.

6. Finite-dimensional vector sublattices

The object of this section is to prove that a Banach lattice which admits a compatible topology necessarily has a large collection of finite-dimensional vector sublattices. (A *vector sublattice* K of L is a linear subspace which is also a sublattice; we say K is a *finite-dimensional vector sublattice* if it is a vector sublattice and is finite-dimensional as a vector space.) We isolate the precise property we need in the following result:

Theorem 6.1. Let L be a Banach lattice which admits a compatible topology. Let $e_1, e_2, \ldots e_N$, $b_1, b_2, \ldots, b_M$ *be positive elements of L such that* $b_j \leqq \sum_{i=1}^N e_i$ *for each j, and let* $\delta > 0$ *be a positive number. Then there is a finite-dimensional vector sublattice K of L and there are positive elements* $e_1^*, \ldots, e_N^*$, $b_1^*, \ldots, b_M^*$ *of K such that*

(1) $0 \leqq e_i^* \leqq e_i$ *and* $\|e_i^* - e_i\| < \delta$ *for each i,*
(2) $0 \leqq b_j^* \leqq b_j$ *and* $\|b_j^* - b_j\| < \delta$ *for each j,*
(3) $\sum_{i=1}^N e_i^*$ *is strictly positive in K.*

It is convenient to first isolate a Lemma. Recall that L is *order complete* if every subset of L^+ which has an upper bound in L actually has a supremum in L.

Lemma 6.2. If the Banach lattice L admits a compatible topology, then L is order complete.

Proof Let Λ be any indexing set and let $\{z_\lambda\}_\Lambda$ be a family of positive elements of L bounded above by the positive element z. Let $\mathscr{F}$ be the set of finite subsets of Λ; for each F in $\mathscr{F}$, set $z_F = \sup\{z_\lambda : \lambda \in F\}$. Since $\mathscr{F}$ is directed by inclusion, the family $\{z_F : F \in \mathscr{F}\}$ is a net of positive elements of L; moreover, $z_F \leqq z$ for each F, so $\{z_F\}$ is a net in the order interval $[0, z]$. By assumption, L admits a Hausdorff vector space topology τ in which the order interval $[0, z]$ is compact. Hence some subnet $\{z_G : G \in \mathscr{J}\}$ converges (in the topology τ) to some element $\bar{z}$ of $[0, z]$. We assert that $\bar{z} = \sup\{z_F\}$.

To see this, we fix an element F_0 of $\mathscr{F}$. The definition of the elements z_F, together with the fact that $\{z_G\}$ is a subnet of $\{z_F\}$, imply that $\{z_G : z_G \geqq z_{F_0}\}$ is