Then $\|z\| \leqq \|w\| < 1$, $\bar{\pi}(z) > \sum(\pi(v_i)/u_i)$, $0 \leqq z^+ = ke \wedge w^+ < ke$ and $0 \leqq z^- = ke \wedge w^- \leqq ke$.

Since $(x_1, \ldots, x_N; \pi)$ is a quasi-equilibrium for $\mathcal{E}$ we have that $\sum x_i = \sum e_i = e$ and $x_i \geqq 0$ for each i. Hence $z^+ \leqq \sum kx_i$ and $z^- \leqq \sum kx_i$. We can use the Riesz Decomposition Property to find vectors $a_1, \ldots, a_N$, $b_1, \ldots, b_N$ in L^+ such that $0 \leqq a_i \leqq kx_i$ and $0 \leqq b_i \leqq kx_i$ for each i, $z^+ = \sum a_i$ and $z^- = \sum b_i$. Notice that $0 \leqq a_i \leqq z^+$ and $0 \leqq b_i \leqq z^-$ for each i, so that $\|a_i - b_i\| \leqq \|z^+ - z^-\| = \|z\| < 1$.

We now define the desired vectors y_i by setting

$$y_i = x_i + \frac{1}{k\mu_i} v_i - \frac{1}{k}(a_i - b_i).$$

We assert that $y_i \in P_i(x_1, \ldots, x_N)$ for each i. This of course follows from the definition of μ_i as the marginal rate of desirability, provided we verify that $(1/k)(a_i - b_i) \leqq x_i + (1/k\mu_i)v_i$ and that $\|(1/k)(a_i - b_i)\| < (1/k\mu_i) \cdot \mu_i$. The first of these inequalities follows from the facts that $a_i \leqq kx_i$, $b_i \geqq 0$ and $v_i \geqq 0$ (for each i); the second follows from the fact that, for each i,

$$\left\|\frac{1}{k}(a_i - b_i)\right\| = \frac{1}{k}\|a_i - b_i\| \leqq \frac{1}{k}\|z\| < \frac{1}{k} = \frac{1}{k\mu_i} \cdot \mu_i$$

since $\|z\| < 1$.

We now consider the cost of the commodity vectors y_i. We cannot estimate these costs individually, but the sum is easy to estimate. We obtain

$$\begin{aligned} \sum \pi(y_i) &= \pi\left(\sum y_i\right) \\ &= \pi\left(\sum\left(x_i + \frac{1}{k\mu_i} v_i - \frac{1}{k}(a_i - b_i)\right)\right) \\ &= \sum \pi(x_i) + \frac{1}{k}\sum \frac{\pi(v_i)}{\mu_i} - \frac{1}{k}\pi(z). \end{aligned}$$

Since $\pi(z) > \sum(\pi(v_i)/\mu_i)$. it follows that $\sum \pi(y_i) < \sum \pi(x_i)$, so that $\pi(y_j) < \pi(X_j) = \pi(e_j)$ for at least one agent j. Since $y_j \in P_j(x_1, \ldots, x_N)$, this violates the assumption that $(x_1, \ldots, x_N; \pi)$ is a quasi-equilibrium. Since we have obtained a contradiction to our supposition that $\sum(\pi(x_i)/\mu_i) < 1$, the proof is complete. □

Remark. For some Banach lattices L, we can actually do better. For example, if L is the Lebesgue space L_1, we can show that $\pi(v_i)/\mu_i \geqq 1$ for some agent i. However, in the general framework, the weaker conclusion of the Price Lemma is the most that is obtainable.