Since $\succsim_t$ is convex, transitive and complete, a standard argument shows that $D(\cdot,\cdot)$ is convex and nonempty valued. We will show that for each fixed $p \in \Delta$, $B(\cdot,p)$ has a measurable graph. To see this for $p \in \Delta$, define $g_p : T \times E \to [-\infty,\infty]$ by $g_p(t,x) = p\cdot x - p\cdot e(t)$. Clearly, $g_p$ is measurable in $t$ and continuous in $x$, and hence by a standard result [see for instance Yannelis (1990, Proposition 3.1)], $g_p(\cdot,\cdot)$ is jointly measurable. Therefore, $g_p^{-1}([-\infty,0]) \in \tau \otimes \beta(E)$. It can be easily checked that

$$\begin{aligned} G_{B(\cdot,p)} &= \{(t,x) \in T \times X(t) : p\cdot x \leq p \cdot e(t)\} \\ &= g_p^{-1}([-\infty,0]) \cap G_X. \end{aligned}$$

Since by assumption $X(\cdot)$ has a measurable graph it follows that for each fixed $p \in \Delta$, $G_{B(\cdot,p)} \in \tau \otimes \beta(E)$. Since $(T,\tau,\mu)$ is a complete measure space and $B(\cdot,\cdot)$ is closed valued, it follows [see for instance Yannelis (1990, Lemma 3.1)] that for each fixed $p \in \Delta$, $B(\cdot,p)$ is lower measurable. Hence, by Castaing's Representation Theorem [see Yannelis (1990)] there exists a family $\{f_n : n = 1,2,\ldots\}$ of measurable functions $f_n : T \to E$ such that for all $t \in T$, $\text{cl}\,\phi\{f_n(t) : t \in T\} = B(t,p)$ (where cl denotes norm closure). For $n = 1,2,\ldots$ let

$$D_n(t,p) = \{y \in B(t,p) : y \succsim_t f_n(t)\}.$$

Since $\succsim_t$ and $B(\cdot,p)$ have measurable graphs so does $D_n(\cdot,p)$. We wish to show that $D(t,p) = \bigcap_{n=1}^{\infty} D_n(t,p)$. Obviously, $D(t,p) \subset D_n(t,p)$ for each $n$, $(n = 1,2,\ldots)$. We now show that $\bigcap_{n=1}^{\infty} D_n(t,p) \subset D(t,p)$. Suppose otherwise, i.e., there exists $z \in \bigcap_{n=1}^{\infty} D_n(t,p)$ and $z \notin D(t,p)$, i.e., there exists $y \in B(t,p)$ such that $y \succsim_t x$. Notice that by assumption the set $\{w \in B(t,p) : w \succsim_t z\}$ is norm closed in $B(t,p)$. Since the family $\{f_n(t) : n = 1,2,\ldots\}$ is norm dense in $B(t,p)$ we can find an $n_0$ such that $f_{n_0}(t) \succsim_t z$, a contradiction. Hence, $D(t,p) = \bigcap_{n=1}^{\infty} D_n(t,p)$ and since for each fixed $p \in \Delta$, $D_n(\cdot,p)$ $(n = 1,2,\ldots)$ have measurable graph so does $D(\cdot,p)$.

Define the excess demand correspondence $\zeta : \Delta \to 2^E$ for the economy $\mathcal{E}$ by

$$\zeta(p) = \int_T D(t,p)d\mu(t) - \int_T e(t)d\mu(t).$$

We must show that $\zeta$ satisfies all the conditions of the Main Lemma. Clearly, for each $p \in \Delta$, $\zeta(p)$ convex valued and $p \cdot \zeta(p) \leq 0$. Since