Theorem 5.1. *Let (T, τ, μ) be a complete, finite, separable measure space, Y be a separable Banach space, P be a metric space and $X : T \to 2^Y$ be an integrably bounded, convex, weakly compact, nonempty valued correspondence. Let $D : T \times P \to 2^Y$ be a nonempty, norm closed, convex valued correspondence such that:*

(i) for all $(t,p) \in T \times P$, $D(t,p) \subset X(t)$,
(ii) for each fixed $t \in T$, $D(t,\cdot)$ is norm u.s.c., and
(iii) for each fixed $p \in P$, $D(\cdot\,,p)$ has a measurable graph.

Then the correspondence $\phi : P \to 2^X$ defined by $\phi(p) = \{x \in S^1_X : x(t) \in D(t,p)$ for almost all $t \in T\}$ is nonempty valued and weakly u.s.c.

Proof. (a) Since for each fixed $p \in P$, $D(\cdot\,,p)$ has a measurable graph and it is nonempty valued, it follows from the Aumann measurable selection theorem that there exists a measurable function $f : T \to Y$ such that $f(t) \in D(t,p)$ for almost all t in T. Since $D(\cdot\,,\cdot)$ is integrably bounded, f is integrable and therefore $f \in \phi(p)$ for each $p \in P$. Hence, ϕ is nonempty valued.

(b) We now show that ϕ is weakly u.s.c. First, notice that by virtue of Diestel's Theorem, S^1_X is compact in the weak topology. Since by assumption (T, τ, μ) is a separable measure space, $L_1(\mu, Y)$ is separable [Dunford-Schwartz (1958, p. 381)]. Since S^1_X is a weakly compact subset of the separable Banach space $L_1(\mu, Y)$, by Theorem V.6.3 in Dunford-Schwartz (1958, p. 334), S^1_X is metrizable. Given that S^1_X with the weak topology is a compact metrizable space, in order to prove that ϕ is weakly u.s.c., it suffices to show that G_ϕ is closed in $P \times S^1_X$, where S^1_X is endowed with the weak topology. To this end let p_n, $(n = 1, 2, \ldots)$ be a sequence in P converging to p (in the metric topology), let y_n, $(n = 1, 2, \ldots)$ be a sequence in S^1_X converging weakly to y, and let $y_n \in \phi(p_n)$. We must show that $y \in \phi(p)$. Let $A_i = \operatorname{con} \bigcup_{j>i} y_j$ for $i = 1, 2, \ldots$. Since y_n converges weakly to y by Mazur's Theorem [see for instance Dunford-Schwartz (1958), Corollary 14, p. 422)] for each $i = 1, 2, \ldots$, there exists a sequence $\{z^i_n\}$ in A_i converging in norm to y. For any $\delta > 0$, we can find n_1 such that $\|z^1_{n_1} - y\| < \delta$. Similarly for $m > 1$, we can find n_m such that $\|z_{n_m} - y\| < \frac{\delta}{m}$. Continuing this process we can construct a sequence, appropriately relabeled $\{z_n\}$, such that $z_n \in \operatorname{con} \bigcup_{j \geq n} y_j$, and z_n converges in norm to y. Without loss of generality we can assume that