$$f: \Delta \to \Delta \text{ by } f(\theta) = \left((\theta_i + \delta_i) \Big/ \left(1 + \sum_{j=1}^{m} \delta_j \right) \right)_{i=1}^{m}.$$

By Brouwer's fixed point theorem, there exists $\theta^* \in \Delta$ such that $\theta^* = f(\theta^*)$, i.e.,

$$\delta_i = \theta_i^* \sum_{j=1}^{m} \delta_j, \quad i = 1, \ldots, m. \tag{6.11}$$

If we show that $\sum_{j=1}^{m} \delta_j = 0$ then by the definition of F_i, $\tilde{y}_i(\theta^*) \in \mathrm{cl}(C + y_i) \cap E_+$ and by continuity (assumption A.12) and the assumption of an extremely desirable commodity (assumption A.10) $\tilde{y}_i(\theta^*) \succsim_t y_i$ for all $t \in T_i$, and all i. Moreover, since $y_i \succ_t x(t)$ for all $t \in T_i$ and for all i by transitivity $\tilde{y}_i(\theta^*)_t\, x(t)$ for all $t \in T_i$ and for all i. Also $\xi \sum_{i=1}^{m} \tilde{y}_i(\theta^*) = \sum_{i=1}^{m} e_i \xi = \int e$. Define $\tilde{y} = \sum_{i=1}^{m} \tilde{y}_i(\theta^*) \chi_{T_i}$ and note that $\int_S \tilde{y} = \int_S e$. Therefore we have found an allocation $\tilde{y}(\cdot)$ which is feasible for the coalition S and preferred to $x(\cdot)$ which in turn was assumed to be in the core of ε, a contradiction. Consequently, we conclude that Claim 6.1 holds. Hence, all that remains to be shown is that $\sum_{j=1}^{m} \delta_j = 0$.

To this end suppose that $\sum_{j=1}^{m} \delta_j > 0$. Notice that by (6.11) we have that $\theta_i^* = 0$ if and only if $\delta_i = 0$. Define J, $K \subset I = \{1, 2, \ldots, m\}$ as follows: $J = \{i \in I : \delta_i = 0\}$, $K = I \backslash J$. Note that $J = \{i \in I : \theta_i^* = 0\}$. Consider any $i \in J$; then by the definition of $\tilde{y}_i(\cdot)$ we have that $\tilde{y}_i(\theta^*) = y_i - u_i$. Now if $u_i \neq 0$, by monotonicity $y_i \succ_t \tilde{y}_i(\theta^*)$ and by virtue of continuity and extreme desirability we can conclude that $\tilde{y}_i(\theta^*) \notin \mathrm{cl}(C + y_i)$. By the definition of F_i, $\delta_i > 0$, a contradiction to the fact that $i \in J$. Hence, $u_i = 0$ for $i \in J$ and so

$$\sum_{i \in I} u_i = \sum_{i \in K} u_i = u. \tag{6.12}$$

Consider any $i \in K$, i.e., $\delta_i > 0$, then by the definition of F_i, it follows that $\tilde{y}_i(\theta^*) = y_i + \theta_i^* w - u_i \notin C + y_i$ for every $i \in K$, and therefore, $\theta_i^* w - u_i \notin C$ for all $i \in K$ which in turn implies that $u_i \notin \theta_i^*(\alpha/\xi) U$ for all $i \in K$. It follows from (6.12), the fact that $u_i \notin \theta_i^*(\alpha/\xi) U$ for all $i \in K$, $\sum_{i \in K} \theta_i^* = 1$ and assumption A.11 that $\sum_{i \in I} u_i = \sum_{i \in K} u_i = u \notin (\alpha/\xi) U$, which contradicts (6.8), (i.e., $u \in (\alpha/\xi) U$). The above contradiction establishes that $\sum_{j=1}^{m} \delta_j = 0$ and this completes the proof of Claim 6.1.

7. Concluding remarks

Remark 7.1. The separability condition on our commodity space E was