$$(\int \phi - \int e) \cap -C = \varnothing. \tag{6.3}$$

Since $-C$ is open it suffices to show that for any $y \in S_\phi$ there exists a sequence $\{(\bar{y}^k, \bar{e}^k): k=1,2,\ldots\}$ in $L_1(\mu, E) \times L_1(\mu, E)$ such that $\bar{y}^k$ converges in the $L_1(\mu, E)$ norm to y, $\int \bar{e}^k \to \int e$, and

$$\int_T \bar{y}^k - \int_T \bar{e}^k \notin -C. \tag{6.4}$$

Let $S = \{t: y(t) >_t x(t)\}$, $S' = T/S$. Without loss of generality we may assume that $\mu(S) > 0$ [for if $\mu(S) = 0$, then $y(t) = e(t)$ μ-a.e. which implies that $\int y - \int e = 0 \notin -C$; consequently (6.3) holds]. In the argument below y and e are restricted to S. Moreover, denote by μ_S the restriction of μ to S. Since $y: S \to E_+$ is Bochner integrable and $>_t$ is norm continuous (assumption A.12) there exist $y_1^k, \ldots, y_{m_k}^k$ in E_+ and T_1^k, $T_2^k, \ldots, T_{m_k}^k$ in τ such that y^k converges in the $L_1(\mu_S, E)$ norm to y, and

$$y^k = \sum_{i=1}^{m_k} y_i^k \chi_{T_i^k} \tag{6.5}$$

$$y_i^k >_t x(t) \quad \text{for} \quad \text{all}\, t \in T_i^k \quad \text{and} \quad \text{all}\, i, i=1,\ldots,m_k, \quad \text{and} \tag{6.6}$$

$$\mu_S(T_i^k) = \xi, \quad i=1,\ldots,m_k, \quad \text{(where } \xi \text{ is a real positive number).} \tag{6.7}$$

Let

$$e^k = \sum_{i=1}^{m_k} \left(\int_{T_i^k} e(t)\, \mathrm{d}\mu(t) \right) \chi_{T_i^k}.$$

Claim 6.1. $\int_S y^k - \int_S e^k \notin -C$.

Assume that Claim 6.1 holds (a proof is given at the end of this section). We can now construct the sequence $\{(\bar{y}^k, \bar{e}^k): k=1,2,\ldots\}$. In particular, define $\bar{y}^k: T \to E_+$ by

$$\bar{y}^k(t) = \begin{cases} y^k(t) & \text{if} \quad t \in S \\ y(t) & \text{if} \quad t \notin S. \end{cases}$$

Similarly define $\bar{e}^k: T \to E_+$ by

$$\bar{e}^k(t) = \begin{cases} e^k(t) & \text{if} \quad t \in S \\ e(t) & \text{if} \quad t \notin S. \end{cases}$$