$$x_i^k(t)=\sum_{i=0}^{k-1}\tilde{x}\left(t+\frac{i}{k}\right)\Big/k\xrightarrow[k\to\infty]{}\int_0^1 x_i(s)\,\mathrm{d}\mu(s)=e_i$$

for almost all t in T; so $x^k(t)$ converges weakly in l_2 to e. Since u is weakly continuous it follows that $u(x^k(t))\to u(e)$ μ-a.e. Notice that by definition, u is bounded. In particular, $u(x)<\pi^2/6$ for every $x\in l_2^+$ [recall the definition of $u(\cdot)$ in (3)] and therefore by the Lebesgue dominated convergence theorem

$$\lim_{k\to\infty}\int_0^1 u(x^k(t))\,\mathrm{d}\mu(t)=\int_0^1\lim_{k\to\infty}u(x^k(t))\,\mathrm{d}\mu(t)=u(e)=u(\textstyle\int e),$$

a contradiction to (5.1). Thus, e is Pareto optimal.

We are now ready to complete the proof of the fact that $C(\varepsilon)=\{e\}$. To this end we first show that

$$C(\varepsilon)\subset\{e\}. \tag{5.2}$$

Suppose that (5.2) is false, then there exists an allocation $x\in C(\varepsilon)$ such that $x(t)\neq e$ for all $t\in S$, $\mu(S)>0$. Let $\tilde{x}=(x+e)/2$, then $\tilde{x}$ is feasible and for all $t\in T$,

$$\begin{aligned}u(\tilde{x}(t))&>\tfrac{1}{2}u(x(t))+\tfrac{1}{2}u(e)\\&\geqq u(e)\end{aligned}$$

[recall that $u(x(t))\geqq u(e)$ for all $t\in T$ since $x\in C(\varepsilon)$]. Moreover, by strict concavity of $u(\cdot)$ we have that

$$u(\tilde{x}(t))>u(e)\quad\text{for}\quad\text{all }t\in S,$$

a contradiction to the fact that e is Pareto optimal.

We now show that

$$\{e\}\subset C(\varepsilon). \tag{5.3}$$

Suppose that (5.3) is false, then there exists a coalition S and an allocation x such that $\int_S x=\int_S e$ and $u(x(t))>u(e)$ for all $t\in S$. Define the allocation $\tilde{x}(\cdot)$ as follows:

$$\tilde{x}(t)=\begin{cases}x(t) & \text{if}\quad t\in S\\ e(t) & \text{if}\quad t\notin S.\end{cases}$$