Example 5.1. Consider the economy $\varepsilon=[(T,\tau,\mu),X,>,e]$ where,

(1) the *space of agents* is $T=[0,1]$, $\tau=$Lebesgue measurable sets, $\mu=$ Lebesgue measure,
(2) the *consumption set* of each agent is, $X(t)=l_2^+$ for all $t\in T$, where l_2 is the space of real sequences (a_n) for which the norm $\|a_n\|=(\sum|a_n|^2)^{1/2}$ is finite,
(3) the *preference relation* of each agent $>_t$, is represented by a strictly concave, monotone weakly continuous utility function, i.e., $u_t(x)=\sum_{i=1}^{\infty} i^{-2}(1-\exp(-i^2x_i))$ for all $t\in T$, and
(4) the *initial endowment* of each agent is $e(t)=e=(1/i^2)_{i=1}^{\infty}$ for all $t\in T$.

We will show that for the above economy, $C(\varepsilon)\neq\varnothing$ and $W(\varepsilon)=\varnothing$. In particular, we will show that the core of ε is unique and consists of the initial endowment e, i.e., $C(\varepsilon)=\{e\}$ and $W(\varepsilon)=\varnothing$. The latter [i.e., $W(\varepsilon)=\varnothing$] will easily follow from the fact that $C(\varepsilon)=\{e\}$. Indeed, since $W(\varepsilon)\subset C(\varepsilon)$, $W(\varepsilon)\subset\{\{e\},\varnothing\}$, but the only candidate as a supporting price p for the allocation e are multiples of $p=(1,1,\dots)$ which are not in the dual of l_2. Hence, all we need to show is that $C(\varepsilon)=\{e\}$.

To prove that $C(\varepsilon)=\{e\}$ we will first need to show that e is Pareto optimal, i.e., there does not exist a feasible allocation x such that $u(x(t))\geqq u(e)$ for all $t\in T$ and $u(x(t))>u(e)$ for all $t\in S$, $S\subset T$, $\mu(S)>0$ (note that the subscript t on u is dropped). To this end suppose by way of contradiction that there exists an allocation x such that $\int_T x=\int_T e\equiv e$, $u(x(t))\geqq u(e)$ for all $t\in T$ and $u(x(t))>u(e)$ for all $t\in S$, $\mu(S)>0$. Without loss of generality we may assume that there exist positive real numbers v, δ, with $u(x(t))\geqq u(e)+v$, $t\in S$, $\mu(S)=\delta$. Extend x to $\tilde{x}$ defined on the interval $[0,1]$ as $\tilde{x}(t)=x(-[t])$, ($[t]=$the integer part of t), and let $x^k(t)=\sum_{i=0}^{k-1}\tilde{x}(t+(i/k))/k$.

Then

$$\int_0^1 u(x^k(t))\,\mathrm{d}\mu(t)=u\left(\sum_{i=0}^{k-1}\tilde{x}\left(t+\frac{i}{k}\right)\Big/k\right)\mathrm{d}\mu(t)$$

$$\geqq\int_0^1\sum_{i=0}^{k-1}(1/k)u\left(\tilde{x}\left(t+\frac{i}{k}\right)\right)\mathrm{d}\mu(t)$$

$$=\int_0^1 u(x(t))\,\mathrm{d}\mu(t)\geqq u(e)+v\delta. \tag{5.1}$$

Notice that each coordinate of $x^k(\cdot)$, denoted by $x_i^k(\cdot)$ (an $L_1[0,1]$ function), converges to e_i μ-a.e. (indeed in L_1), so