$$\int_S \tilde{y} = \int_S y + v = \int_T y - \int_{S'} e + v = \int_T e - \int_{S'} e = \int_S e \quad \text{[recall (4.5)]}.$$

Therefore, we have found an allocation $\tilde{y}(\cdot)$ which is feasible for the coalition S and is also preferred to the allocation x, which in turn was assumed to be in the core of ε, a contradiction which establishes the validity of (4.2).

We may now separate the set $\mathrm{cl}(\int \phi - \int e) = \mathrm{cl}\int \phi - \int e$ from $\mathrm{int}\, E_-$. Clearly the set $\mathrm{int}\, E_-$ is convex and non-empty. We wish to show that $\mathrm{cl}\int \phi - \int e$ is convex and non-empty as well. Observe first that by the definition of $\phi(\cdot)$, 0 is an element of $\int \phi - \int e$ and this shows that $\mathrm{cl}\int \phi - \int e$ is non-empty. Since (T, τ, μ) is atomless (assumption A.2) by Theorem 1 in Khan (1985) or Theorem 4.2 in Hiai and Umegaki (1977), $\mathrm{cl}\int \phi$ is convex. Thus, by Theorem 9.10 in Aliprantis and Burkinshaw (1985, p. 136) there exists a continuous linear functional $p \in E^* \setminus \{0\}$, $p \geqq 0$ such that

$$p \cdot y \geqq p \cdot \int e \quad \text{for} \quad \text{all } y \in \int \phi. \tag{4.6}$$

Since by assumption A.6, $>_t$ has a measurable graph, so does ϕ, i.e., $G_\phi \in \tau \otimes \beta(E_+)$. Therefore, it follows from Theorem 2.2 in Hiai and Umegaki (1977) that

$$\inf_{y \in \int \phi} p \cdot y = \int \inf_{z \in \phi} p \cdot z \geqq \int p \cdot e. \tag{4.7}$$

It follows from (4.7) that

$$\mu\text{-a.e. } p \cdot z \geqq p \cdot e(t) \quad \text{for} \quad \text{all } z >_t x(t). \tag{4.8}$$

To see this, suppose that for $z \in \phi(\cdot)$, $p \cdot z < p \cdot e(t)$ for all $t \in S$, $\mu(S) > 0$.

Define the function $\tilde{z}\colon T \to E_+$ by

$$\tilde{z}(t) = \begin{cases} z(t) & \text{if} \quad t \in S \\ e(t) & \text{if} \quad t \notin S. \end{cases}$$

Obviously, $\tilde{z} \in \phi(\cdot)$. Moreover,

$$\begin{aligned} \int_T p \cdot \tilde{z} &= \int_S p \cdot z + \int_{T \setminus S} p \cdot e \\ &< \int_S p \cdot e + \int_{T \setminus S} p \cdot e = \int p \cdot e, \end{aligned}$$

a contradiction to (4.7).