Proof of Theorem 3.1. It suffices to prove that $\bigcap_{i=1}^n F(x_i) \neq \emptyset$ for every finite subset $\{x_1, \ldots, x_n\}$ of X. Suppose otherwise; i.e., $\bigcap_{i=1}^n F(x_i) = \emptyset$ for some finite subset $\{x_1, \ldots, x_n\}$ of X. Let Δ be the simplex spanned by the finite set $\{x_1, \ldots, x_n\}$. Since the topology induced on any finite dimensional subspace by the topology of Y is equivalent to the Euclidean topology, Δ is homeomorphic to a Euclidean ball [Kelley and Namioka (1963, Theory 7.3, p. 49)]. Denote by d the Euclidean metric in the finite dimensional subspace spanned by $\{x_1, \ldots, x_n\}$. Define $\psi : \Delta \to 2^\Delta$ by $\psi(x) = \{y \in \Delta : x \notin F(y)\}$. Then for each $x \in \Delta$, $\psi(x) \neq \emptyset$. Indeed at least one x_i, $(1 \leq i \leq n)$ is in $\psi(x)$, for otherwise $x \in \bigcap_{i=1}^n F(x_i)$. For each $y \in \Delta$ let $\psi^{-1}(y) = \{x \in \Delta : y \in \psi(x)\} = \Delta \setminus \{x \in \Delta : x \in F(y)\}$. Since $\{x \in \Delta : x \in F(y)\} = \Delta \cap F(y)$ is closed in Δ, the set $\psi^{-1}(y)$ is open in Δ for each $y \in \Delta$. Define $\phi : \Delta \to 2^\Delta = \operatorname{con} \psi(x)$. Then ϕ is convex and nonempty valued. Moreover, ϕ has open lower sections in Δ [Yannelis and Prabhakar (1983, Lemma 5.1)]. Nonempty valueness of ϕ implies that for every $x \in \Delta$ there is a $y \in \Delta$ such that $x \in \phi^{-1}(y)$. Hence, the collection $\{\phi^{-1}(y) : y \in \Delta\}$ is an open cover of Δ. But Δ compact implies that there exists a finite set of points $\{y_1, \ldots, y_n\}$ such that $\Delta \subseteq \bigcup_{i=1}^n \phi^{-1}(y_i)$. Define $\alpha_i : \Delta \to \mathbb{R}_+$ by $\alpha_i(x) = d(x, \Delta \setminus \phi^{-1}(y_i))$, $1 \leq i \leq n$. Set $g_i(x) = (\alpha_i(x))/(\sum_{j=1}^n \alpha_j(x))$ for all $x \in \Delta$, $1 \leq i \leq n$. Then, $g_i(x) = 0$ for $x \notin \phi^{-1}(y_i)$, $0 \leq g_i(x) \leq 1$ and $\sum_{i=1}^n g_i(x) = 1$ for all $x \in \Delta$. Define $f : \Delta \to \Delta$ by $f(x) = \sum_{i=1}^n g_i(x) y_i$. Clearly f is continuous and for each i such that $g_i(x) \neq 0$, $x \in \phi^{-1}(y_i)$ or $y_i \in \phi(x)$. Hence, $f(x)$ is a convex combination of points y_i in the convex set $\phi(x)$ and so $f(x) \in \phi(x)$ for all $x \in \Delta$. By Brouwer's fixed point theorem there exists $x^* \in \Delta$ such that $x^* = f(x^*) \in \phi(x^*) = \operatorname{con} \psi(x^*)$. But $x^* \in \operatorname{con} \psi(x^*)$ implies that there exist points $y_1, \ldots, y_m$ in Δ and real numbers $a_1, \ldots, a_m$, $a_j \geq 0$, $(1 \leq j \leq m)$, $\sum_{j=1}^m a_j = 1$ such that $x^* = \sum_{j=1}^m a_j y_j$ and $y_j \in \psi(x^*)$ for all j. But $y_j \in \psi(x^*)$ implies that $x^* \notin F(y_j)$ for all j, a contradiction to assumption (i). Therefore $\bigcap_{i=1}^n F(x_i) \neq \emptyset$, and this completes the proof.

References

Aliprantis, C. D. and Burkinshaw, O., 1985, *Positive Operators*, Academic Press, New York.