(5.1) there exists $y \in F$ such that $y \in P_i(x)$ for all i, or
(5.2) for at least one agent i, $e \in P_i(x)$.

For each $i \in I$ define $\psi_i : X \to 2^X$ by $\psi_i(x) = \text{con}\, P_i(x)$. Since by assumption (4.3) P_i has τ-open lower sections it follows from Lemma 5.1 in Yannelis and Prabhakar (1983, p. 239), that ψ_i has τ-open lower sections in X. Let $\psi_{i|F}$ be the restriction of ψ_i to F. It follows from (5.1) that:

(5.3) for all $x \in F$ there exists $y \in F$ such that $y \in P_i(x) \subset \text{con}\, P_i(x) = \psi_{i|F}(x)$ for all $i \in I$.

For each $i \in I$ define $\Phi_i : F \to 2^F$ by $\Phi_i(x) = \psi_{i|F}(x) \cap F$. Define $A = \{w \in F : \text{there exist } z \in F \text{ such that } z \in P_i(w) \text{ for all } i \in I\}$. It can be easily checked that A is open in F. It follows from (5.3) that:

(5.4) for all $x \in A$, $\Phi_i(x)$ is nonempty for all $i \in I$.

Notice that from assumption (4.2) we have that $x \notin \text{con}\, \Phi_i(x) = \Phi_i(x)$ for all $x \in F$. Moreover, it can be easily seen that Φ_i has τ-open lower sections in F, i.e., Φ_i is of class $\mathcal{L}$. For $x \in F$, let $S_x = \{i \in I : e \in P_i(x)$. It follows from (5.2) that

(5.5) for all $x \in F$ and all $i \in S_x$, $\Phi_i(x) \neq \emptyset$.

Indeed, from (5.2) we can conclude that for all $x \in F$ and all $i \in S_x$, $e \in P_i(x) \subset \text{con}\, P_i(x) = \psi_{i|F}(x)$. Consequently, for all $x \in F$ and all $i \in S_x$, $e \in \Phi_i(x)$.

Define the correspondence $\theta : F \to 2^F$ by

$$\theta(x) = \begin{cases} \bigcap_{i \in I} \Phi_i(x) & \text{if } x \in A \\ \bigcap_{i \in S_x} \Phi_i(x) & \text{if } x \in F \setminus A. \end{cases}$$

It follows from (5.4) and (5.5) that

(5.6) for all $x \in F$, $\theta(x) \neq \emptyset$.

Notice that F is nonempty, convex, bounded and τ-closed. Moreover, F lies on the order interval $[0, Ne]^N = \{x \in X : 0 \leq x_j \leq Ne \text{ for all } j \in I\}$ which is τ-compact. Therefore, F is a τ-compact subset of X. If we show that θ is $\mathcal{L}$-majorized we can then appeal to Corollary 3.1. To this end let $x \in F$. Then either (a) $x \in A$, or (b) $x \in F \setminus A$. If (a) holds then $\theta(x) = \bigcap_{i \in I} \Phi_i(x) \neq \emptyset$. Choose $w \in \theta(x)$. Then $w \in \Phi_i(x)$ for all i. Fix an agent j in I. Since A is an open set in F, and Φ_j has τ-open lower