7. Induced Cycles in the Complement of the Star Graph

We establish in this section the other part of the proof of Theorem 4, namely that the complement $H^c = (V, F)$ of the star graph does not have induced cycles of length greater than 4.

Lemma 13. H^c does not contain an induced cycle of length greater than 4.

Proof: Since H does not contain an induced 5 cycle, and the complement of a 5 cycle is a 5 cycle, H^c cannot contain an induced 5 cycle. Assume to the contrary that there exists an induced cycle $C = (V', F')$ in H^c, with $|V'| > 5$. For convenience, let $V' = \{1, 2, \cdots, n\}, n > 5$, and let $<i, i+1 \bmod n> \in H'$. Since $1 \not\wedge 2$, we have the constriction $Cons\,(1,2)$. Since edge $<i, j> \notin F$, for $j \neq i+1 \bmod n$, we have that $i \wedge j$.

We now assert that $4, 5, \cdots, n-1$ cannot be in $Cons\,(1,2)$. Suppose 4 were in $Cons\,(1,2)$. $4 \wedge 1$ and $4 \wedge 2$, and $5 \wedge 1$ and $5 \wedge 2$. From Lemma 9, $4 \wedge 5$, a contradiction. Similar arguments establish that $5, \cdots, n-1$ cannot be in $Cons\,(1,2)$. It is further clear that $Cons\,(1,2)$ is not of type III, else $4, 5, \cdots, n-1$ have to be inside $Cons\,(1,2)$ in order to see 1 and 2. Suppose $Cons\,(1,2)$ is of type II. Then, by Lemma 11, $4 \wedge 5$, a contradiction.

We now have that all of $Cons\,(i, i+1 \bmod n)$ must be of type I. Let us now further classify type I constrictions as type IA (where the two dent lines are vertical) and type IB (where the two dent lines are horizontal). We now assert that $Cons\,(1,2)$ and $Cons\,(i, i+1)$ cannot both be of type IA or IB (IA, say), for some $i \in \{4, \cdots, n-2\}$. To see this, we reason as follows: Neither the i dent line nor the $i+1$ dent line of $Cons\,(i, i+1)$ can be inside $Cons\,(1,2)$, since i and $i+1$ see both 1 and 2. Now, let the i dent line be outside $Cons\,(1,2)$, implying that $Cons\,(i, i+1)$ and $Cons\,(1,2)$ are on different sides of the i dent line. This would imply that that $i+1$ does not indirectly see 1 and 2, a contradiction. A simple combinatorial argument now shows that it is impossible to obtain an induced 7 cycle using only type IA and IB constrictions for cycle edges. Thus, let $n = 6$.

It is clear that at least two of $Cons\,(1,2)$, $Cons\,(3,4)$ and $Cons\,(5,6)$ must be of the same type (IA or IB). Assume, without loss of generality, that $Cons\,(1,2)$ and $Cons\,(3,4)$ are both of type IA. We now assert that this would imply that $Cons\,(2,3)$ must be of type IA and that $Cons\,(1,2)$ is contained in $Cons\,(3,4)$ (see Figure 13 (a)). Thus, the only way that one could possibly obtain an induced 6 cycle is by using type IA constrictions for three consecutive edges and type IB consttrictions for the other three edges. Let edges $<1,2>$, $<2,3>$, and $<3,4>$ use type IA constrictions and let $<4,5>$, $<5,6>$ and $<6,1>$ use type IB constrictions. The arrangement is shown in Figures 13 (a) and 13 (b).