2 Core, Value, and Fair Allocations

2.1 Core Allocations

DEFINITION 2.1.1 : An allocation $x \in X$ is a **core allocation** for $\mathcal{E} = \{(X_i, u_i, e_i) : i \in I\}$ if

(1) $\sum_{i \in I} x_i = \sum_{i \in I} e_i$

(2) There does not exists a coalition $S \subset I$ and $(x'_i)_{i \in S} \in \Pi_{i \in S} X_i$ such that $\sum_{i \in S} x'_i = \sum_{i \in S} e_i$ and $x'_i \succ_i x_i, \forall i \in S$.

and denote by $C(\mathcal{E})$ be the set of all core allocations for $\mathcal{E}$.[7]

N.B. We can replace the second conditon with (2′) There is no coalition $S \subset I$ and $(x'_i)_{i \in S} \in \Pi_{i \in S} X_i$ such that $\sum_{i \in S} x'_i = \sum_{i \in S} e_i$ and $x'_i \succeq_i x_i, \forall i \in S$ and $x'_i \succ_i x_i, \exists i \in S$. But this condition is less reasonable.

N.B. If an allocation is individually rational and pareto optimal for two agents, then it is the core. The core says that the coalition of a single agent or the grand coalition of two cannot block, and individual rationality says that a singleton coalition cannot improve upon, and grand coalition cannot either. Generally, the set of core allocations is a subset of the set of individually rational and Pareto optimal allocations.

N.B. : Even if one agent has whole endowment of the economy and the others have zero, it is still a Pareto optimum but not fair. The core depends on the initial endowments but the Pareto optimum does not.

THEOREM 2.1.1 : $\boldsymbol{C}(\mathcal{E}) \subset \boldsymbol{WP}(\mathcal{E})$.

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$$C(\mathcal{E}) = V(I) \setminus \bigcup_{S \subset I} intV(S),$$

where

$$\begin{aligned} V(S) &= \{w \in R^{|S|} : \exists (x_i)_{i \in S} \in \Pi_{i \in S} X_i \text{ such that } \sum_{i \in S} x_i = \sum_{i \in S} e_i \text{ and } w_i \leq u_i(x_i), \forall i \in S\}, \\ intV(S) &= \{w \in R^{|S|} : \exists (x_i)_{i \in S} \in \Pi_{i \in S} X_i \text{ such that } \sum_{i \in S} x_i = \sum_{i \in S} e_i \text{ and } w_i < u_i(x_i), \forall i \in S\}. \end{aligned}$$