that $x_i' \succeq_i x_i$. By the nonsatiation of $\succeq_1$, $\exists \hat{x}_1 \in X_1$ such that $\hat{x}_1 \succ_1 x_1$. By the semi-strict convexity of $\succeq_1$, $x_1^\alpha := \alpha \hat{x}_1 + (1-\alpha)x_1 \succ_1 x_1, \forall \alpha \in (0,1)$. Let $z = x_1^\alpha + x_i' + \sum_{k \in I \setminus \{1,i\}} x_k$. Then $z \in C(x)$, which implies that $p \cdot x_1^\alpha + p \cdot x_i' \geq p \cdot x_1 + p \cdot x_i$. By letting $\alpha \to 0$ we get $p \cdot x_i' \geq p \cdot x_i$.
To complete the proof, we must show that $x_1^o \sim_1 x_1 \Rightarrow p \cdot x_1^o \geq p \cdot x_1$. Suppose that $x_1^o \sim_1 x_1$. By the nonsatiation of $\succeq_1$, $\exists \bar{x}_1 \in X_1$ such that $\bar{x}_1 \succ_1 x_1^o$. By the convexity of $\succeq_1$, $x_1^\lambda := \lambda \bar{x}_1 + (1-\lambda)x_1^o \succ_1 x_1^o, \forall \lambda \in (0,1)$. Since $x_1^o \sim_1 x_1$, $x_1^\lambda \succ_1 x_1$ so that $p \cdot x_1^\lambda \geq p \cdot x_1$ by the previous result. By letting $\lambda \to 0$, we get $p \cdot x_1^o \geq p \cdot x_1$. □

THEOREM 1.7.12 (Second Welfare Theorem II) : Let $\succeq_i$ be complete, transitive, convex, and nonsatiated for every i. Then a weakly Pareto optimal allocation can be supported by some $p \in R^\ell \setminus \{0\}$.

PROOF: Let $C_i(x_i) := \{x_i' \in X_i : x_i' \succ_i x_i\}, \forall i \in I$. Then $C(x) := \sum_{i \in I} C_i(x_i)$ is nonempty and convex since the preferences are complete, transitive, semi-strictly convex and nonsatiated. Note that $e := \sum_i e_i \notin C(x)$ since x is weakly Pareto optimal. By the separating hyperplane theorem, $\exists p \in R^\ell \setminus \{0\}$ such that $\forall z \in C(x), p \cdot z \geq p \cdot e = p \cdot \sum_{i \in I} x_i$. Now, for each $i \in I$, suppose that $x_i' \succeq_i x_i$. By the nonsatiation of $\succeq_k$, $\exists \hat{x}_k \in X_k$ such that $\hat{x}_k \succ_k x_k, \forall k \in I \setminus \{i\}$. By the semi-strict convexity of $\succeq_k$, $x_k^\alpha := \alpha \hat{x}_k + (1-\alpha)x_k \succ_k x_k, \forall k \in I \setminus \{i\}, \forall \alpha \in (0,1)$. Similarly, we get $x_i^\alpha = \alpha \hat{x}_i + (1-\alpha)x_i' \succ_i x_i' \succeq_i x_i$ with $\hat{x}_i \succ_i x_i'$. Then $\sum_i x_i^\alpha \in C(x)$, which implies that $p \cdot \sum_i x_i^\alpha \geq p \cdot \sum_i x_i$. By letting $\alpha \to 0$ we get $p \cdot x_i' \geq p \cdot x_i$. □

THEOREM 1.7.12 : $W(\mathcal{E}) \cap Q(\mathcal{E}) \subset P(\mathcal{E})$.

COROLLARY 1.7.13 : For all i, there is at most one satiation consumption and his preference is locally nonsatiated at the nonsatiated consumptions. Then $W(\mathcal{E}) \subset P(\mathcal{E})$.

1.7.3 Optimality of Quasi-Equilibria

LEMMA 1.7.14 : Suppose that X_i is convex and $\succeq_i$ is continuous. Suppose that (x_i, p) is such that $x_i' \succ x_i$ implies $p \cdot x_i' \geq p \cdot x_i$ and $p \cdot x_i > \inf p \cdot X_i$. Then $x_i' \succ_i x_i$ implies $p \cdot x_i' > p \cdot x_i$.

THEOREM 1.7.15 : Let X_i be convex and $\succeq_i$ be continuous. A quasi-equilibrium allocation x with $p \cdot x_i > \inf p \cdot X_i$ for some i is a weakly Pareto optimal.