there exists a feasible $x' \in X$ such that $x'_i \succ_i x_i, \forall i \in I$. Since x is a Walrasian equilibrium, $p \cdot x'_i > p \cdot e_i, \forall i \in I$. Thus $p \cdot \sum_{i \in I} e_i = \sum_{i \in I} p \cdot x'_i > \sum_{i \in I} p \cdot e_i$, which is a contradiction. □

EXAMPLE 1.7.2 : A Walrasian equilibrium allocation need not be Pareto optimal.

THEOREM 1.7.9 (First Welfare Theorem III) : If $\succeq_i$ is reflexive and strictly convex for every i, $\boldsymbol{W}(\mathcal{E}) \subset \boldsymbol{P}(\mathcal{E})$.

PROOF: Take a Walrasian equilibrium x with respect to p. Suppose it is not Pareto optimal. Then there exists a feasible $x' \in X$ such that $x'_i \succeq_i x_i, \forall i \in I$ and $x'_i \succ_i x_i, \exists i \in I$. Since x is a Walrasian equilibrium, $p \cdot x'_i > p \cdot e_i \geq p \cdot x_i, \exists i \in I$. But $p \cdot \sum_{i \in I} e_i = \sum_{i \in I} p \cdot x'_i = \sum_{i \in I} p \cdot x_i$ Thus $p \cdot x'_k < p \cdot x_k, \exists k \in I$. Moreover, $x'_k \succeq_k x_k, x_k \succeq_k x_k$ and $x'_k \neq x_k$. By the strict convexity of preferences, $x^{\alpha}_k \succ_k x_k$ so that $p \cdot x^{\alpha}_k > p \cdot e_k$. By letting $\alpha \to 1$, we have $p \cdot x'_k \geq p \cdot e_k$, a contradiction. □

THEOREM 1.7.10 (First Welfare Theorem IV) : If $\succeq_i$ is transitive and strictly monotonic for every i, $\boldsymbol{W}(\mathcal{E}) \subset \boldsymbol{P}(\mathcal{E})$.

PROOF: Take a Walrasian equilibrium x with respect to p. Suppose it is not Pareto optimal. Then there exists a feasible $x' \in X$ such that $x'_i \succeq_i x_i, \forall i \in I$ and $x'_i \succ_i x_i, \exists i \in I$. Since x is a Walrasian equilibrium, $p \cdot x'_i > p \cdot e_i \geq p \cdot x_i, \exists i \in I$. But $p \cdot \sum_{i \in I} e_i = \sum_{i \in I} p \cdot x'_i = \sum_{i \in I} p \cdot x_i$ Thus $p \cdot x'_k < p \cdot e_k, \exists k \in I$. However, for every $\varepsilon \in R^{\ell}_{+} \setminus \{0\}$, $x'_k + \varepsilon \succ_k x'_k \succeq_k x_k$ so that $p \cdot (x'_k + \varepsilon) \geq p \cdot e_k$. By letting $\varepsilon \to 0$, we have $p \cdot x'_k \geq p \cdot e_k$, a contradiction. □

THEOREM 1.7.11 (Second Welfare Theorem I) : If $\succeq_i$ is complete, transitive and semi-strictly convex for every i and $\succeq_1$ is nonsatiated, a Pareto optimal allocation can be supported by some $p \in R^{\ell} \setminus \{0\}$.

PROOF: Let $C^o_1(x_1) := \{x'_1 \in X_1 : x'_1 \succ_1 x_1\}$ and $C_i(x_i) := \{x'_i \in X_i : x'_i \succeq_i x_i\}, \forall i \in I \setminus \{1\}$. Then $C(x) := C^o_1(x_1) + \sum_{i=2}^{n} C_i(x_i)$ is nonempty and convex since $\succeq_1$ is nonsatiated and $\succeq_i$ is complete, transitive, and semi-strictly convex for every i. Note that $e := \sum_i e_i \notin C(x)$ since x is Pareto optimal. By the separating hyperplane theorem, $\exists p \in R^{\ell} \setminus \{0\}$ such that $\forall z \in C(x), p \cdot z \geq p \cdot e = p \cdot \sum_{i \in I} x_i$. This implies that $z_1 \succ_1 x_1 \Rightarrow p \cdot z_1 \geq p \cdot x_1$ and $z_i \succeq_i x_i \Rightarrow p \cdot z_i \geq p \cdot x_i, \forall i \in I \setminus \{1\}$. Indeed, if $x'_1 \succ_1 x_1$, then $z = x'_1 + \sum_{i \in I \setminus \{1\}} x_i \in C(x)$, from which it follows that $p \cdot x'_1 \geq p \cdot x_1$. On the other hand, for each $i \in I \setminus \{1\}$, suppose