PROOF: Since x is a quasi-equilibrium, it is feasible, i.e., $\sum_{i \in I} x_i = \sum_{i \in I} e_i$. The reflexivity implies that $p \cdot x_i = p \cdot e_i$ for every i. Let $x_i' \succ_i x_i$. Then $p \cdot x_i' \geq p \cdot e_i$. Now suppose $p \cdot x_i' = p \cdot e_i$. Since $e_i \in int X_i$, there is $\hat{x}_i \in X_i$ such that $p \cdot \hat{x}_i < p \cdot e_i$. Consider $x_i^\alpha = \alpha x_i' + (1-\alpha)\hat{x}_i$. By continuity, $x_i^\alpha \succ_i x_i$ for α close to 1. Therefore, $p \cdot x_i^\alpha \geq p \cdot e_i$. But by construction, $p \cdot x_i^\alpha < p \cdot e_i$. This is a contradiction. □

THEOREM 1.7.4 : For all i, there is at most one satiation consumption and his preference is locally nonsatiated at the nonsatiated consumptions. Then $W(\mathcal{E}) \subset Q(\mathcal{E})$.

THEOREM 1.7.5 : $\boldsymbol{P}(\mathcal{E}) \subset \boldsymbol{WP}(\mathcal{E})$.
PROOF: By definition. □

THEOREM 1.7.6 : If $\succeq_i$ is strictly monotonic and continuous for every i, $\boldsymbol{P}(\mathcal{E}) = \boldsymbol{WP}(\mathcal{E})$.
PROOF: Pick $x \in \boldsymbol{WP}(\mathcal{E}) \setminus \boldsymbol{P}(\mathcal{E})$. Then $\exists x'$ such that $x_i' \succeq_i x_i, \forall i \in I$ and $x_k' \succ_k x_k, \exists k \in I$. By the continuity of preferences, $\exists \varepsilon \in R_+^\ell \setminus \{0\}$ such that $X_k \ni x_k' - \varepsilon \succ_k x_k$. Let $x_i^o := x_i' + \varepsilon/(n-1) \in X_i, \forall i \in I \setminus \{k\}$ and $x_k^o := x_k' - \varepsilon$. Then x^o is feasible. However, by the strict monotonicity of preferences, $x_i^o \succ_i x_i, \forall i \in I$, which contradicts the weak Pareto optimality of x. □

1.7.2 Optimality of Equilibria

THEOREM 1.7.7 (First Welfare Theorem I) : Let $\succeq_i$ be reflexive, continuous, and monotonic for every i. If a feasible allocation x is supported by a price $p \in R^\ell \setminus \{0\}$ such that $p \cdot \sum e_i \neq 0$, then x is weakly Pareto optimal.

PROOF: Suppose otherwise. There exists a feasible $x' \in X$ such that $x_i' \succ_i x_i, \forall i \in I$. Since p supports x, $p \cdot x_i' \geq p \cdot x_i, \forall i \in I$. Since $p \cdot \sum_{i \in I} e_i = \sum_{i \in I} p \cdot x_i' = \sum_{i \in I} p \cdot x_i$, we conclude that $p \cdot x_i' = p \cdot x_i, \forall i \in I$. However, by the continuity of preferences, $\exists \alpha \in (0,1)$ such that $(1-\alpha)x_i' \succ_i x_i, \forall i \in I$ and, by the p-supportability of x, $(1-\alpha)p \cdot x_i' \geq p \cdot x_i, \forall i \in I$. Thus $(1-\alpha)p \cdot x_i' \geq p \cdot x_i'$. Since $p \cdot x_i' \geq 0$, $(1-\alpha)p \cdot x_i' = p \cdot x_i', \forall i \in I$. This implies that $p \cdot x_i' = 0, \forall i \in I$ so that $p \cdot \sum_{i \in I} e_i = 0$, which is a contradiction. □

THEOREM 1.7.8 (First Welfare Theorem II) : $\boldsymbol{W}(\mathcal{E}) \subset \boldsymbol{WP}(\mathcal{E})$.

PROOF: Take a Walrasian equilibrium x. Suppose it is not weakly Pareto optimal. Then