1.2 Gale-Debreu-Nikaido Lemma

THEOREM 1.2.1 (Gale-Debreu-Nikaido) : Let $Z : \Delta \to 2^{R^\ell}$ be an excess demand correspondence satisfying the following conditions:

(1) Z is nonempty-valued, compact-valued, convex-valued, and upper hemi-continuous,

(2) for every $p \in \Delta, \exists z \in Z(p)$ such that $p \cdot z \le 0$.

Then, $\exists p^* \in \Delta$ such that $Z(p^*) \cap R^\ell_- \neq \emptyset$.

PROOF: Suppose otherwise, i.e., $\forall p \in \Delta$, $Z(p) \cap R^\ell_- = \emptyset$. Since $Z(p)$ is nonempty compact convex and R^ℓ_- is nonempty closed convex, by the separating hyperplane theorem, $\exists q^* \in R^\ell \setminus \{0\}$ such that $\sup_{y \in R^\ell_-} q^* \cdot y < \inf_{z \in Z(p)} q^* \cdot z$. Note that $\sup_{y \in R^\ell_-} q^* \cdot y = 0$ so that $q^* \cdot z > 0, \forall z \in Z(p)$. Without loss of generality, we can take $q^* \in \Delta$.
Define $F : \Delta \to 2^\Delta$ by $F(p) := \{q \in \Delta : q \cdot z > 0, \forall z \in Z(p)\}$. Then, we want to show that F is nonempty, convex valued and lower-hemicontinuous, so that we can apply the Michael selection theorem.
Since for every $p \in \Delta$, $q^* \cdot z > 0, \forall z \in Z(p)$, $q^* \in F(p)$, i.e., F is nonempty valued. Pick q_1, q_2 in $F(p)$. Then for every $p \in \Delta, q_1 \cdot z > 0$ and $q_2 \cdot z > 0, \forall z \in Z(p)$. Thus for every $p \in \Delta$ and for every $\alpha \in [0,1]$, $(\alpha q_1 + (1-\alpha) q_2) \cdot z > 0, \forall z \in Z(p)$, which implies that $\alpha q_1 + (1-\alpha) q_2 \in F(p)$, i.e., F is convex valued.
For each $q \in \Delta$,

$$\begin{aligned} F^{-1}(q) &= \{p \in \Delta : q \in F(p)\} && (1) \\ &= \{p \in \Delta : q \cdot z > 0, \forall z \in Z(p)\} && (2) \\ &= \{p \in \Delta : Z(p) \subset \{z : q \cdot z > 0\}\}. && (3) \end{aligned}$$

Since $V := \{z : q \cdot z > 0\}$ is open and Z is upper-hemicontinuous, $\forall q \in \Delta, F^{-1}(q)$ is open in Δ, *i.e.*, F has open lower sections. Thus, F is lower-hemicontinuous[6].
By the Michael selection theorem, there exists a continuous function $f : \Delta \to \Delta$ such that $f(p) \in F(p)$ for all $p \in \Delta$. This function maps points from the nonempty compact convex set into itself. Therefore, it fulfills the condition of the Brouwer's fixed point theorem.

[6]Note that for every open subset W of Δ, $\bigcup_{q \in W} F^{-1}(q) = \{p \in \Delta : F(p) \cap W \neq \emptyset\}$. In fact, $p' \in \cup_{q \in W} F^{-1}(q)$ iff $p' \in F^{-1}(q)$ for some $q \in W$ iff $q \in F(p'), q \in W$ iff $q \in F(p') \cap W$ iff $F(p') \cap W \neq \emptyset$ iff $p' \in \{p \in \Delta : F(p) \cap W \neq \emptyset\}$. Since the union of open sets is open, $\{p \in \Delta : F(p) \cap W \neq \emptyset\}$ is open for every open subset W of Δ.