(13) $\succeq$ is **proper** at x if there exists a $v \in R_+^\ell \setminus \{0\}$ and a neighborhood V of zero such that $z \in R^\ell$ and $x - \alpha v + z \succeq x$ with $\alpha \in R_+$ implies $z \notin \alpha V$.

(14) $\succeq$ is **uniformly proper** if it is proper at every $x \in X$.

N. B We can define the convexities of preference $\succeq$ in the following forms.

(i) $x' \succeq x$ implies $\alpha x' + (1-\alpha)x \succeq x$ for every $\alpha \in [0,1]$.

(ii) $x' \succ x$ implies $\alpha x' + (1-\alpha)x \succ x$ for every $\alpha \in (0,1]$.

(iii) $x' \sim x$ with $x' \neq x$ implies $\alpha x' + (1-\alpha)x \succ x$ for every $\alpha \in (0,1)$.

THEOREM 1.1.3 We assume that $\succeq$ is complete and transitive.

(1) If $\succeq$ continuous. Then (iii) implies (ii), which,in turn, implies (i).

(2) (i) holds for every $x \in X$ iff $\succeq$ is convex.

(3) $\succeq$ is convex iff $\succ$ is convex.

(4) If $\succeq$ is convex, continuous, strictly monotone, then (ii) holds.

(5) If $\succeq$ is continuous, then (iii) holds iff $\succeq$ is strictly convex.

THEOREM 1.1.4 A preference $\succeq$ is proper iff there is a non-trivial open cone $\Gamma \in R^\ell$ such that for every $x \in X$,

$$\Gamma \cap (-R_+^\ell) \neq \emptyset, \; (\{x_i\} + \Gamma) \cap R(x_i) = \emptyset.$$

N. B There is a commodity which is very desirable in the sense that its marginal rates of substitution with respect to any other commodity are uniformly bounded above. It is always satisfied by monotone preference. It can be equivalently formulated as : there is a non-trivial open cone Γ such that

$$(\{x_i\} + \Gamma) \cap X_i \subset P(x_i)$$

for every i and $x_i \in X_i$.

THEOREM 1.1.5 Every uniformly proper vector is extremely desirable.

THEOREM 1.1.6 If a preference $\succeq$ is monotonic and has extremely desirable bundle, then it is uniformly proper.