DEFINITION 9.9: Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be two measurable spaces. $f : \Omega \to \Omega'$ is $(\mathcal{F}, \mathcal{F}')$ **-measurable** if $f^{-1}(A) \in \mathcal{F}, \forall A \in \mathcal{F}'$.

DEFINITION 9.10: Let $f : \Omega \to R$. f is **measurable** with respect to $\mathcal{F}$ (or $\mathcal{F}$-measurable) if $f^{-1}(A) \in \mathcal{F}, \forall A \subset \mathcal{B}(R)$.

NOTE A **random variable** is a real-valued measurable function in probability space.

DEFINITION 9.11: Let X_i's be random variables. $\sigma(X_1, \ldots, X_n)$ is the smallest σ-field with respect to which $X_1, \ldots, X_n$ are measuarble.

N. B. $\sigma(X_1, \ldots, X_n) = \bigvee_{i=1}^{n} \sigma(X_i)$.

Example : Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\mathcal{F} = 2^{\Omega}$. A consumer has a random endowments : $e(\omega_1) = 1, e(\omega_2) = 0, e(\omega_3) = 0$. Then e is measurable with respect to $\sigma(e) = \{\{\omega_1\}, \{\omega_2, \omega_3\}, \Omega, \emptyset\}$

DEFINITION 9.12: Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Let f be a nonnegative measurable simple function, i.e., $f = \sum_{i=1}^{n} a_n \mathbf{1}_{A_i}$ where $a_i \in \mathbf{R}_+, \forall i = 1, \ldots, n$ and $(A_1, \ldots, A_n)$ be a finite measurable partition of Ω. The **integral** of f on Ω is defined by

$$\int f d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$$

DEFINITION 9.13: Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. The **integral** of a nonnegative random variable f on Ω is defined by

$$\int f d\mu = \sup\{\int f' d\mu : f' \text{ is simple and } f' \leq f\}$$

DEFINITION 9.14: Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $A_1, \ldots, A_n$ be a finite measurable partition Ω. The **integral** of a random variable f on Ω is defined by

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu,$$

where $f = f^+ - f^-$ and $f^+ = f \vee 0, f^- = (-f) \vee 0$.

Monotone Convergence Theorem : Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and (f_n) be a sequence of measurable functions on Ω.

$$0 \leq f_n \leq f_{n+1}, \forall n \in \mathbf{N} \text{ and } f_n \to f, \mu\text{-}a.e. \Rightarrow \int f_n d\mu \to \int f d\mu.$$