THEOREM 4.11 (Product): Let $\varphi_i : X \to 2^{Y_i}$ be a correspondence for every $i \in I$. Define the correspondence $\prod_{i\in I} \varphi_i : X \to 2^Y$ by $(\prod_{i\in I} \varphi_i)(x) = \prod_{i\in I} \varphi_i(x)$, where $Y = \prod_{i\in I} Y_i$.

(1) If φ_i has open (closed) graph for every $i \in I$, so does $\prod_i \varphi_i$.

(2) If φ_i is compact-valued and upper hemi-continuous at x for every $i \in I$, so is $\prod_i \varphi_i$.

(3) If φ_i is lower hemi-continuous at x for $i = 1, \ldots, n$, so is $\prod_i \varphi_i$.

THEOREM 4.12 (Sum): Let $Y_i \subset R^\ell$ and $\varphi_i : X \to 2^{Y_i}$ be a correspondence for $i = 1, 2, \ldots, n$. Define the correspondence $\sum_{i=1}^m \varphi_i : X \to 2^Y$ by $(\sum_{i=1}^m \varphi_i)(x) = \sum_{i=1}^m \varphi_i(x)$, where $Y = \sum_i Y_i$.

(1) If φ_i is compact-valued and upper hemi-continuous at x for $i = 1, \cdots, n$, so is $\sum_i \varphi_i$.

(2) If φ_i is lower hemi-continuous at x for $i = 1, \ldots, n$, so is $\sum_i \varphi_i$.

(3) If φ_i has open (closed) graph for $i = 1, \ldots, n$, so does $\sum_i \varphi_i$.

THEOREM 4.13 (Convex Hull): Let Y be convex. Let $\varphi : X \to 2^Y$ be a correspondence. Define the correspondence $co\varphi : X \to 2^Y$ by $(co\varphi)(x) = co[\varphi(x)]$.

(1) If φ is compact-valued and upper hemi-continuous at x, so is $co\varphi$.

(2) If φ is lower hemi-continuous (has open graph, has open lower sections) at x, so is (does) $co\varphi$.

EXAMPLE 4.4: Consider a correspondence $\varphi : R \to 2^R$.

$$\varphi(x) = \begin{cases} \{0, 1/x\} & \text{if } x \neq 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

Here, φ has a closed graph but $co\varphi$ does not have a closed graph.

5 Maximum Theorem

THEOREM 5.1(Berge): Let $\varphi : X \to 2^Y$ be a continuous correspondence with nonempty, compact-valued, and suppose $f : Gr\varphi \to \Re$ is continuous. Then