(4) Let A and B be disjoint nonempty convex subsets of R^ℓ. Let A and B be closed. Then A and B can be strictly separated by a hyperplane.

(5) Let A and B be disjoint nonempty convex subsets in R^ℓ. Then there exists a hyperplane separating the sets A and B.

N. B. In (2), if z is on the boundary of S, the hyperplane is called **supporting hyperplane**.

DEFINITION 3.8: Let X be a convex subset of R^ℓ and let $f : X \to R$ be a function.

(1) A function f is **concave** if for $x, x' \in X$, $f(\alpha x + (1-\alpha)x') \geq \alpha f(x) + (1-\alpha)f(x')$ for $\alpha \in [0,1]$.

(2) A function f is **convex** if $-f$ is concave.

(3) A function f is **quasi-concave** if $\{x \in X : f(x) \geq \alpha\}$ is convex for every $\alpha \in R$.

(4) A function f is **quasi-convex** if $(-f)$ is quasi-concave.

4 Correspondences

A correspondence is a set-valued function and arise naturally in many economic applications, for instance, budget correspondence, excess demand correspondence, etc. The biggest difference between functions and correspondences has to do with the definitions of an inverse image. The inverse image of a set A under a function f is the set $\{x : f(x) \in A\}$. For a correspondence, there are two reasonable generalizations, the upper inverse and the lower inverse. Having two definitions of the inverse leads to two definitions of continuity, that is, lower hemicontinuity and upper hemicontinuity.

Let $X \subset R^\ell$ and $Y \subset R^m$.

DEFINITION 4.1: A **correspondence** $\varphi : X \to 2^Y$ is a function from X to the family of all subsets of Y.

A correspondence $\varphi : X \to 2^Y$ is compact-valued (nonempty-valued, convex-valued, open-valued, closed-valued, bounded-valued) if $\varphi(x)$ is a compact (nonempty, convex, open, closed, bounded) subset of Y for every $x \in X$.