(8) $co(\sum_i S_i) = \sum_i coS_i$.

N. B. It follows from (6) that coS is the smallest convex set containing S.

EXAMPLE 3.1: Consider

$$S = \{(x, y) \in R^2 : y \geq \frac{1}{|x|}\}.$$

Then S is closed, but S is not compact and coS is not closed.

DEFINITION 3.5: A **hyperplane** in R^ℓ is a set $\{x \in R^\ell : p \cdot x = \alpha\}$ where $p \in R^\ell \setminus \{0\}$ and $\alpha \in R$. We denote it by $H(p, \alpha)$. The vector p is **normal** to the hyperplane $H(p, \alpha)$.

N. B. A hyperplane is the set of solutions of one linear equation in ℓ variables.

DEFINITION 3.6: A set $\{x : p \cdot x \leq \alpha\}$ is a **closed lower half space** $H(p, \alpha)$. A set $\{x : p \cdot x < \alpha\}$ is an **open lower half space** $H(p, \alpha)$.

DEFINITION 3.7: Two sets A and B in R^ℓ are **separated** by a hyperplane $H(p, \alpha)$ if there are $p \in R^\ell \setminus \{0\}$ and $\alpha \in R$ such that for every $x \in A$ and $y \in B$

$$p \cdot x \leq \alpha \leq p \cdot y$$

They are **strictly separated** if the inequalities are replaced by strict inequalities. They are **strongly separated** if $\sup_{x \in A} p \cdot x < \alpha < \inf_{y \in B} p \cdot y$.

THEOREM 3.6 (Separating Hyperplane Theorem):

(1) Let S be a nonempty closed and convex subset in R^ℓ and $z \notin S$. Then there exists a point $x^* \in S$ and a hyperplane $H(p, \alpha)$ through x^* such that

$$p \cdot z < \alpha = p \cdot x^* = \inf_{x \in S} p \cdot x.$$

(2) Let S be a nonempty convex set of R^ℓ and $z \notin S$. Then there exists a hyperplane $H(p, \alpha)$ through z such that for every $x \in S$

$$p \cdot z = \alpha \leq p \cdot x$$

(3) Let A and B be disjoint nonempty convex subsets of R^ℓ. Let A be closed and B be compact. Then A and B can be strongly separated by a hyperplane.