3 Convex Sets

DEFINITION 3.1: A subset S of R^ℓ is **convex** if for $x, x' \in S$, $x^\alpha = \alpha x + (1-\alpha)x' \in S$ for every $\alpha \in [0,1]$.

DEFINITION 3.2: x is an **extreme point** if $x = \alpha x' + (1-\alpha)x''$ with $\alpha \in (0,1)$ implies $x = x' = x''$.

DEFINITION 3.3: $\sum_{i=1}^{m} \alpha x_i$ is a **(finite) convex combination** of $x_1, x_2, \ldots, x_m$ if $\alpha_1, \alpha_2, \ldots, \alpha_m$ satisfies $\sum_{i=1}^{m} \alpha_i = 1$ and $\alpha_i \geq 0$ for every i. A **strict positive convex combination** is a convex combination where $\alpha_i > 0$ for every i.

DEFINITION 3.4: The **convex hull** of $S \subset R^\ell$ is the set of all finite convex combinations from S and is denoted by coS.

THEOREM 3.1 (Carathéodory): Let S be a subset of R^ℓ. Then, every point $x \in coS$ is a convex combination of $\ell + 1$ points in S.

THEOREM 3.2 (Krein-Milman): Let S be a nonempty compact convex subset of R^ℓ. Then $S = co(exS)$ where exS is the set of extreme points.

THEOREM 3.3 (Shapley-Folkman): Let S_i be nonempty subsets of R^ℓ for every $i = 1, 2, \ldots, m$. For every $x \in co(\sum_{i=1}^{m} S_i)$, there exist $x_i \in coS_i$, $i = 1, 2, \ldots, m$ such that $x = \sum_i x_i$ and $\#\{i : x_i \notin S_i\} \leq \ell$.

THEOREM 3.4:

(1) If S_i is convex for every $i \in I$, so is $\bigcap_{i \in I} S_i$.

(2) If S_i is convex for each $i = 1, \ldots, m$, so are $\sum_i S_i$ and $\prod_i S_i$.

(3) Let $\alpha \in R$. If S is convex, so is αS.

(4) If S is convex, so are $intS$ and $\bar{S}$.

(5) If S is open (compact), so is coS.

(6) $coS := \bigcap\{C \subset R^\ell : C \text{ is convex and } S \subset C\}$.

(7) $co\bar{S} \subset \bar{co}S$