is the **boundary** of S and is denoted by ∂S. **N. B.** By the symmetry of the definition, $\partial S = \partial(S^c)$. Also, a simple argument shows that $\partial S = \bar{S} \cap \bar{S}^c$.

DEFINITION 2.15: A subset S of R^ℓ is **bounded** if there are two points x' and x'' in R^ℓ such that $x' \leq x \leq x''$ for every $x \in S$.

THEOREM 2.4: A bounded sequence has a convergent subsequence.

DEFINITION 2.16: Let $X \subset R^\ell$ and $Y \subset R^m$. A function $f : X \to Y$ is **continuous at** $x \in X$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f(x), f(x')) < \varepsilon$ whenever $d(x, x') < \delta$. A function f is **continuous** on X if f is continuous at every point of X.

DEFINITION 2.17: A family of subsets $\{A_i : i \in I\}$ of R^ℓ is a **cover** of $S \subset R^\ell$ if $S \subset \bigcup_{i \in I} A_i$. If a subfamily of $\{A_i : i \in I\}$ also covers S, then it is a **subcover**. Any cover of S consisting of open sets is an **open cover** of S.

DEFINITION 2.18: A subset S of R^ℓ is **compact** if every open cover of S can be reduced to a finite subcover.

THEOREM 2.5: Let S be a subset of R^ℓ. The following are equivalent :

(1) S is compact.

(2) S is closed and bounded.

(3) Every sequence of S has a convergent subsequence whose limit belongs to S.

(4) Every infinite subset of S has an accumulation point in S.

(5) Every collection of closed subsets of S with the finite intersection property (*i.e.*, every finite subcollection has a nonempty intersection) has a nonempty intersection.

THEOREM 2.6:

(1) Every closed subset of a compact set is compact.

(2) If $f : X \to Y$ is continuous, and K is compact in X, then $f(K)$ is compact in Y.

(3) S_i is compact for every $i \in I$ iff $\prod_{i \in I} S_i$ is compact.

(4) S_i is compact for every $i = 1, \ldots, m$ iff $\sum_{i=1}^m S_i$ is compact.