(2) $d(x, y) = 0$ iff $x = y$.

(3) $d(x, y) = d(y, x)$.

(4) $d(x, y) + d(y, z) \geq d(x, z)$.

The pair (X, d) is called a **metric space**.

Given a metric d, let $B_\varepsilon(x) = \{y : d(x, y) < \varepsilon\}$, the **open ε-ball** around x. A set U is open in the **metric topology** generated by d if for each point x in U there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$. A topological space is **metrizable** if there exists a metric d on X generating the topology of X.

The **Euclidean metric** on $\Re^n$, $d(x, y) = \{\sum_{i=1}^n (x_i - y_i)^2\}^{1/2}$, defines its usual topology, called **Euclidean topology**.

The metric, a real-valued function, allows us to analyze spaces using what we know about the real numbers. The distinguishing features of the theory of the metric spaces, which are absent from the theory of topology, are the notions of *uniform continuity* and *completeness*.

DEFINITION 2.2: For a nonempty subset A of a metric space (X, d), its **diameter** is defined by diam $A = sup\{d(x, y) : x, y \in A\}$. A set A is **bounded** if diam $A < \infty$.

THEOREM 2.1(Heine-Borel) : Subsets of $\Re^n$ are compact if and only if they are closed and bounded.

From now on, let the space be R^n (Euclidean space).

DEFINITION 2.3: Define $R^\ell = \{(x_1, x_2, \ldots, x_\ell) : x_i \in R, i = 1, 2, \ldots, \ell\}$ and let $x \in R^\ell$ and $y \in R^\ell$.

(1) $x \leq y$ means $x_i \leq y_i$ for every $i = 1, \ldots, \ell$.

(2) $x < y$ means $x_i \leq y_i$ and $x \neq y$.

(3) $x \ll y$ means $x_i < y_i$ for every $i = 1, \ldots, \ell$.

(4) $R_+^\ell = \{x \in R^\ell : x \geq 0\}$.

(5) $R_{++}^\ell = \{x \in R^\ell : x \gg 0\}$.