NOTE: In a metric space, continuity at a point x reduces to the familiar $\varepsilon - \delta$ definition: For every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$.

THEOREM 1.1: For a function $f : X \to Y$ between topological spaces, the following are equivalent :

(1) f is continuous on X.

(2) If C is an closed subset of Y, then $f^{-1}(C)$ is an closed subset of X.

(3) For every subset B of X, $f^{-1}(intB) \subset int[f^{-1}(B)]$.

(4) For every subset A of X, $f(\bar{A}) \subset \overline{f(A)}$.

DEFINITION 1.4: An **open cover** of a set K is a collection of open sets whose union includes K. A subset K of a topological space is **compact** if every open cover of K includes a finite subcover. That is, K is compact if every family $\{V_i : i \in I\}$ of open sets satisfying $K \subset \bigcup_{i \in I} V_i$ has a finite subfamily $V_{i_1}, V_{i_2}, \cdots, V_{i_n}$ such that $K \subset \bigcup_{j=1}^{n} V_{i_j}$.

THEOREM 1.2: Every continuous function between topological spaces carries compact sets to compact sets.

Proof: Let $f : X \to Y$ be a continuous function between two topological spaces, and let K be a compact subset of X. Also, let $\{V_i : i \in I\}$ be an open cover of $f(K)$. Then $\{f^{-1}(V_i) : i \in I\}$ is an open cover of K. By the compactness of K, there exists $i_1, \ldots, i_n$ satisfying $K \subset \bigcup_{j=1}^{n} f^{-1}(V_{i_j})$. Hence,

$$f(K) \subset f(\bigcup_{j=1}^{n} f^{-1}(V_{i_j})) = \bigcup_{j=1}^{n} f(f^{-1}(V_{i_j})) \subset \bigcup_{j=1}^{n} V_{i_j}$$

which shows that $f(K)$is a compact subset of Y.

Corollary (Weierstrass) : A continuous real-valued function defined on a compact space achieves its maximum and minimum values.

2 Metric Space

DEFINITION 2.1: A **metric** on a set X is a function $d : X \times X \to \Re$ satisfying:

(1) $d(x, y) \geq 0$.