Restrict P_t to U_t and notice that it satisfies all conditions of the Caratheodory Selection Theorem in [11, Theorem 4.1]. Hence, there exists a function $f_t : U_t \to Y$ such that $f_t(\omega, \tilde{x}_{-t}) \in P_t(\omega, \tilde{x}_{-t})$ for all $(\omega, \tilde{x}_{-t}) \in U_t$. Moreover, for each $\omega \in \Omega$, $f_t(\omega, \cdot)$ is continuous on the set $U_t^\omega = \{\tilde{x}_{-t} \in L_{X_{-t}} : P_t(\omega, \tilde{x}_{-t}) \neq \phi\}$ and for each $\tilde{x}_{-t} \in L_{X_{-t}}$, $f_t(\cdot, \tilde{x}_{-t})$ is measurable on the set $U_t^{\tilde{x}_{-t}} = \{\omega \in \Omega : P_t(\omega, \tilde{x}_{-t}) \neq \phi\}$. Also, by Proposition 3.1 in [11], $f_t(\cdot, \cdot)$ is jointly measurable.

For each $t \in T$, define $F_t : \Omega \times L_{X_{-t}} \to 2^Y$ by

$$F_t(\omega, \tilde{x}_{-t}) = \begin{cases} \{f_t(\omega, \tilde{x}_{-t})\} & \text{if } (\omega, \tilde{x}_{-t}) \in U_t \\ Y & \text{otherwise.} \end{cases}$$

By Lemma 2.12 in [14], for each $\tilde{x}_{-t} \in L_{X_{-t}}$, $F_t(\cdot, \tilde{x}_{-t})$ is lower measurable. Since $F_t(\cdot, \cdot)$ is closed–valued, we can conclude that for each $\tilde{x}_{-t} \in L_{X_{-t}}$, $F_t(\cdot, \tilde{x}_{-t})$ has a measurable graph. Clearly for each $(\omega, \tilde{x}_{-t}) \in \Omega \times L_{X_{-t}}$, $F_t(\omega, \tilde{x}_{-t})$ is non-empty.

By Lemma 6.1 in [13], for each $\omega \in \Omega$, $F_t(\omega, \cdot)$ is (weakly) u.s.c.[2] For each $t \in T$ define $F_t : L_{X_{-t}} \to 2^{L_X}$ by

$$F_t(\tilde{x}_{-t}) = \{y \in L_X : \quad y_t(\omega) \in F_t(\omega, \tilde{x}_{-t}) \ \ \mu\text{–a.e.}\}.$$

Since for each $\tilde{x}_{-t} \in L_{X_{-t}}$, $F_t(\cdot, \tilde{x}_{-t})$ has a measurable graph by virtue of the Aumann measurable selection theorem, there exists an $\mathcal{F}_t$–measurable function $g_t : \Omega \to Y$ such that $g_t(\omega) \in F_t(\omega, \tilde{x}_{-t})$ μ–a.e. Since for each $(\omega, \tilde{x}_{-t}) \in \omega \times L_{X_{-t}}$, $F_t(\omega, \tilde{x}_{-t}) \subset X_t(\omega)$ and $X_t(\cdot)$ is integrably bounded, it follows that $g_t \in L_{X_t}$. Hence, $g_t \in F_t(\tilde{x}_{-t})$, i.e., F_t is non-empty–valued. By Diestel's theorem[3] L_{X_t} is a weakly compact subset of $L_1(\mu, Y)$.

Since the weak topology for a weakly compact subset of a separable Banach space is metrizable [4, p. 434], we can conclude that L_{X_t} is metrizable, and since T is countable, so is L_X. It follows from the Fatou Lemma in infinite–dimensional spaces (see, for example, [12]) that for each $t \in T$, $F_t(\cdot)$ is (weakly) upper semicontinuous, and it is obviously convex and closed–valued. Define $\phi : L_X \to 2^{L_X}$ by

$$\Phi(x) = \prod_{t \in T} F_t(\tilde{x}_{-t}).$$

Clearly, L_X is compact, convex, non-empty, and Φ is a (weakly) u.s.c. convex, closed, non-empty–valued correspondence from L_X to 2^{L_X}. By the Fan–Glicksberg fixed point theorem there exists $\tilde{x} \in L_X$ such that $x^* \in \Phi(x^*)$. One

[2] i.e., for each $\omega \in \Omega$, the set $\{\tilde{x}_{-t} \in L_{X_{-t}} : F_t(\omega, \tilde{x}_{-t}) \subset V\}$ is open in $L_{X_{-t}}$ (where each L_{X_s} $(s \neq t)$ is endowed with the weak topology) for every (norm) open subset V of Y.

[3] For a proof using James' theorem on weak compactness, see [12, Theorem 3.1].