Fuzzy Sets towards Forming Boolean Algebra

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Abstract

This paper aims to revisit the main issues which were hindering the fuzzy sets to form a Boolean algebra and efforts have been made to show that fuzzy sets do form Boolean algebra if the extended definition of complementation of fuzzy sets using reference function is considered.

Keywords: *fuzzy membership value, fuzzy membership function.*

1. Introduction

The mathematical system developed by the English mathematician George Boole (1815-1864) in an attempt to give symbolic form to Aristotle's system of logic became known as Boolean algebra. This system codified several rules of relationship between mathematical quantities limited to one or two possible values: true or false, 1 or 0.In other words, the distinguishing feature of Boolean algebra is that it deals with the study of binary variables. As a result of which it is considered that the laws of Boolean algebra often differ from the laws of real number algebra. It is believed since the inception of fuzzy sets theory that fuzzy sets cannot satisfy all the properties of Boolean algebra. Here we shall discuss about the initial conception of fuzzy set complementation and wish to replace it with some newer proposals.

According to fuzzy set theory, the belonging of an element to a fuzzy set is gradual or characterized by membership or belonging function. If Ω is defined as the set of objects of x, a fuzzy set A in Ω is defined as:

$$A=\{(x,\mu_A(x),x\epsilon\Omega\}$$

where $\mu_A(x)$ is called the membership function of the fuzzy set A and the complementation is defined by

$$\mu_{A^c}(x)=1-\mu_A(x)$$

Dubious and Prade [1], were in the view that the most debatable properties of sets when considering fuzzy sets are law of contradiction (a proposition cannot be both true and false) and excluded middle (a proposition either true or false): $A\cap A^c=\varphi$ and$A\cup A^c=\Omega$. Since a fuzzy set has a gradual boundary ,it seemed to them that there should be some overlap between a set A and its complement A^c and that together they cannot fill the whole universe. According to them choosing min –max system with complementation one minus the membership function of the set is the only way of preserving all the properties of Boolean structure except these two laws. According to them fuzzy sets cannot form Boolean algebra since the gradual membership and compositionality of membership grades are *incompatible* Boolean structure. Furthermore, it was mentioned that although the law of contradiction and excluded middle fail, they hold to a limited extent since $(A \cap A^c)_{\alpha=}\emptyset$ for $\alpha > 0.5$ and $(A \cap A^c)_{\alpha=}\Omega$ for $\alpha < 0.5$

But at the same time it was mentioned that keeping the excluded middle and contradiction laws implies the deletion of some important properties like idempotence and mutual distributivity.

So it is obvious from the statement that acceptance of those two properties would lead to the deletion of other two properties, so according to them it was never possible for fuzzy sets to form Boolean algebra.

2. Properties of Boolean Algebra

Boolean algebra can be defined as a set of subsets A,B,C ⊂ X which satisfies the following properties:

i. $A \cup A^c = \Omega$, $A \cap A^c = \varphi$
ii. $A \cup B = B \cup A, A \cap B = B \cap A$
iii. $A \cup (B \cup C) = (A \cup B) \cup C,$
iv. $A \cap (B \cap C) = (A \cap B) \cap C$
v. $A \cup (A \cap B) = A, A \cap (A \cup B) = A$
vi. $A \cup \emptyset = A, A \cap \Omega = A$
vii. $A \cup A^c = \Omega, A \cap A^c = \emptyset$
viii. $(A^c)^c = A$
ix. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
x. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
xi. $(A \cup B)^C = A^c \cap B^c$
xii. $(A \cap B)^c = A^c \cup B^c$

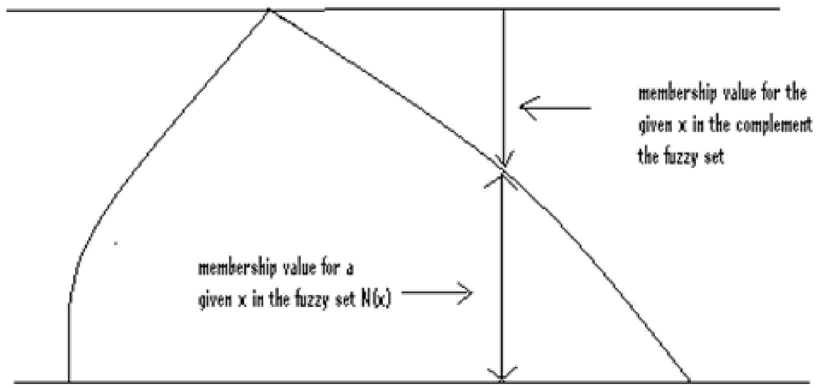
In this article, an attempt is made to show that unlike crisp sets fuzzy sets also satisfy those two properties without hampering the properties of idempotence and mutual distributivity as will be seen later and thus it can form Boolean algebra. But the beliefs that were there since inception of fuzzy set theory cannot be rejected so easily without mentioning any appropriate mathematical framework. So here lies the need of a mathematical tool which can serve our purpose by establishing our claim. To do so, we require the help of the definition of complementation of fuzzy sets on the basis of reference function proposed by Baruah [2].This might be the only one available method on which we can rely on to meet our purpose.

3. Baruah"s Definition of Extended Fuzzy Sets

It is observed that in Zadehian definition of complementation, membership value and membership function had been taken to be of the same meaning. Baruah [2] has derived the complementation of fuzzy sets on the basis of the assumption that the membership value and membership function should be different in the sense that for usual fuzzy sets membership value and membership functions are of course equivalent. Consequently a fuzzy number is defined with the help of two functions: a fuzzy membership function $\mu_1(x)$ and a reference function $\mu_2(x)$ so that $\mu_1(x)$ is to be counted from $\mu_2(x)$, both $\mu_1(x)$ and $\mu_2(x)$ lying between 0 and 1 such that $\mu_1(x) \geq$

$\mu_2(x)$ Then for a fuzzy number denoted by {x, $\mu_1(x)$, $\mu_2(x)$,x$\epsilon\Omega$} , the membership value for any x is defined as $\{\mu_1(x) - \mu_2(x)\}$. It is to be worth mentioning fact that the concept of reference function has been derived with the help of superimposition of sets.

This can be represented in the following manner:



The above figure depicts what is said in favor of complementation of fuzzy sets in a very simple and clear form.

According to the concepts mentioned above it can be said that a normal fuzzy number $N=[\alpha,\beta,\gamma]$ defined with a membership function $\mu_N(x)$, where

$$\mu_N(x)=\psi_1(x) \text{ if } \alpha\le x\le\beta$$
$$=\psi_2(x) \text{ if } \beta\le x\le\gamma$$
$$=0, \text{ otherwise}$$

and $\boldsymbol{\psi}_1(x)$ is continuous and nondecreasing in the interval $[\alpha,\beta]$, and $\boldsymbol{\psi}_2(x)$ is continuos and nonincreasing in the interval $[\beta,\gamma]$ with

$$\psi_1(\alpha)=\psi_2(\gamma)=0 \text{ and}$$
$$\psi_1(\beta)=\psi_2(\beta)=1,$$

the complement N^c will have the membership function

$$\mu_N c(x)=1\ ,\ -\infty<x<\infty$$

but there is a condition that $\mu_N c(x)$ is to be counted from $\psi_1(x)$, if $\alpha\le x\le\beta$, from $\psi_2(x)$, if $\beta\le x\le\gamma$ and from zero otherwise, so that to keep a difference between the fuzzy membership function and the fuzzy membership value. As a consequence of the results obtained above, which, it is defined that the fuzzy membership function of the normal fuzzy number N to be equal to 1 for entire real line, with membership value counted from the membership value of N. This concept of complementation is called the extended definition of complementation of fuzzy sets

In accordance with the process discussed above, a fuzzy set defined by

$$A=\{x,\mu(x),x\epsilon\Omega\}$$

would be defined in this way as

$$A=\{x,\mu(x),0,x\epsilon\Omega\}$$

so that the complement would become

$$A^c = \{x, 1, \mu(x), x\epsilon\Omega\}$$

4. Union and Intersection of Extended Fuzzy Sets

The union and intersection of two fuzzy sets two fuzzy sets

$$A(\mu_{1,}\mu_2) = \{\mathrm{x}, \mu_1(x), \mu_2(x), x\epsilon\Omega\}$$

$$\mathrm{B}(\mu_{3,}\mu_4) = \{x, \mu_3(x), \mu_4(x), x\epsilon\Omega\}$$

is defined in the following manner:

$$A(\mu_{1,}\ \mu_2) \cup \mathrm{B}(\mu_{3,}\ \mu_4) = \{x, \max(\mu_1(x), \mu_3(x)), \min\ (\mu_2(x), \mu_4(x));\ x\epsilon\Omega\}$$

$$A(\mu_{1,}\mu_2) \cap \mathrm{B}(\mu_{3,}\ \mu_4) = \{x, \min(\mu_1(x), \mu_3(x)), \max(\mu_2(x), \mu_4(x));\ x\epsilon\Omega\}$$

The extended definition of complementation of fuzzy sets using reference function which is mentioned above leads us to conclude that for any fuzzy set A we have the following results:

$$A \cup A^c = \text{the universal set } \Omega \text{ and}$$

$$A \cap A^c = \text{the null set } \emptyset$$

Thus we see that the two laws which were not accepted to hold for fuzzy sets do hold if the complementation is defined in this proposed manner.

With the help of some numerical examples we shall show how the extended definition of fuzzy sets also satisfy some of the properties which were considered to be not applicable in case of fuzzy swts till recently.

5. Numerical Examples

Let us consider three fuzzy sets as

A={(3,1),(4,.2),(5,.3),(6,.4),(7,.6),(8,.8),(10,1),(12,.8),(14,.6)}
B={(2,.4),(3,.6),(4,.8),(5,1),(6,.8),(7,.6,0),(8,.4)}
C= {(2,.4),(4,.8),(5,1),(7,.6)}

which would take the following form if extended definition is used

A={(3,1,0),(4,.2,0),(5,.3,0),(6,.4,0),(7,.6,0),(8,.8,0),(10,1,0),(12,.8,0),(14,.6,0)}
B={(2,.4,0),(3,.6,0),(4,.8,0),(5,1,0),(6,.8,0),(7,.6,0),(8,.4,0)}
C={(2,.4,0),(4,.8,0),(5,1,0),(7,.6,0)}

Then the complement of the above sets is denoted in the following manner according to the extended definition:

A^c={(3,1,1),(4,1,.2),(5,1,.3),(6,1,.4),(7,1,.6),(8,1,.8),(10,1,1),(12,1,.8),(14,1,.6)}
B^c={(2,1,.4),(3,1,.6),(4,1,.8),(5,1,1),(6,1,.8),(7,1,.6),(8,1,.4)}
C^c={(2,1,.4),(4,1,.8),(5,1,1),(7,1,.6)}

Thus we get the following results:

$A \cup B$={2,.4,0),(3,1,0),(4,.8,0),(5,1,0),(6,.8,0),(7,.6,0),(8,.8,0),(10,1,0),(12,.8,0),(14,.6,0)}
$(A \cup B)^c$={(2,1,.4),(3,1,1),(4,1,.8),(5,1,1),(6,.1,.8),(7,1,.6),(8,1,.8),(10,1,1),(12,1,.8),(14,1,.6)}
$A^c \cap B^c$={(2,1,.4),(3,1,1),(4,1,.8),(5,1,1),(6,1,.8),(7,1,.6),(8,1,.8),(10,1,1),(12,1,.8),(14,1,.6)}
$A \cap B$={(2,0,0),(3,.6,0),(4,.2,0),(5,.3,0),(6,.4,0),(7,.6,0),(8,.4,0),(10,0,0),(12,0,0),(14,0,0)}
$(A \cap B)^c$={(2,1,0),(3,1,.6),(4,1,.2),(5,1,.3),(6,1,.4),(7,1,.6),(8,1,.4),(10,1,0),(12,1,0),(14,1,0)}
$A^c \cup B^c$={(2,1,0),(3,1,.6),(4,1,.2),(5,1,.3),(6,1,.4),(7,1,.6),(8,1,.4),(10,1,0),(12,1,0),(14,1,0)}
$A \cup C$={(2,.4,0),(3,1,0),(4,.8,0),(5,1,0),(6,.4,0),(7,.6,0),(8,.8,0),(10,1,0),(12,.8,0),(14,.6,0)}
$B \cap C$={(1,0,0),(2,.4,0),(3,0,0),(4,.8,0),(5,1,0),(6,0,0),(7,.6,0),(8,0,0)}
$A \cup (B \cap C$ ={(1,0,0),(2,.4,0),(3,1,0),(4,.8,0),(5,1,0),(6,.4,0),(7,.6,0),(8,.8,0),(10,1,0),(12,.8,0),(14,.6,0)}
$(A \cup B) \cap (A \cup C)$={(2,.4,0),(3,1,0),(4,.8,0),(5,1,0),(6,.4,0),(7,.6,0),(8,.8,0),(10,1,0),(12,.8,0),(14,.6,0)}
$B \cup C$={(2,.4,0),(3,.6,0),(4,.8,0),(5,1,0),(6,.8,0),(7,.6,0),(8,.4,0)}
$A \cap C$={(2,0,0),(3,0,0),(4,.2,0),(5,.3,0),(6,0,0),(7,.6,0),(8,0,0),(10,0,0),(12,0,0),(14,0,0)}
$A \cap (B \cup C)$ ={(2,0,0),(3,.6,0),(4,.2,0),(5,.3,0),(6,.4,0),(7,.6,0),(8,.4,0),(10,0,0),(12,0,0),(14,0,0)}
$(A \cap B) \cup (A \cap C)$={((2,0,0),(3,.6,0),(4,.2,0),(5,.3,0),(6,.4,0),(7,.6,0),(8,.4,0),(10,0,0),(12,0,0),(14,0,0)}

Thus from the above discussion, we see that the extended fuzzy sets satisfy the properties:

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

This part is discussed to yield a deeper understanding of the principle involved and the principle is discussed and this is similar to those obtained for crisp sets. In other words, the above result shows that in the view of extended definition of complementation of a fuzzy set in the basis of reference function, fuzzy sets may be viewed as a crisp set. We can apply this notion of complementation which satisfies all the properties of crisp sets to reach our destination. The above results indicate that the reasons which were put forward in favors of fuzzy sets for not forming Boolean algebra cannot be taken into considerations any more. The difficulties stem out from the failure to obey such laws which created a rather unsatisfactory situation no longer exist. Hence we can say that all the properties required for the formation would be satisfied by fuzzy sets if the complementation is defined in this way. So we can claim that fuzzy sets do form Boolean algebra from what was explained above.

6. Conclusion

It had been accepted that fuzzy sets do not form Boolean algebra and this difficulty stem out from the fact that given a fuzzy set neither its intersection with its complement is the null set nor its union with the complement is the universal set. In this article efforts are made to show that fuzzy sets fuzzy set satisfy those two laws which are required for the formation of Boolean algebra without interfering with the other properties such as idempotence and mutual distributivity. So it is concluded that fuzzy sets do form Boolean algebra. But it is a point to be noticed here that the complementation of fuzzy sets on the basis of reference function played a crucial role in reaching the conclusion.

References

[1] Didier Dubious and Henry Prade: Fundamentals of Fuzzy Sets, Springer 2000

[2] Hemanta. K. Baruah, Fuzzy Membership with respect to a Reference Function ,Journal of the Assam Science Society, Vol.40,No.3,1999,65-73

[3] Hemanta .K. Baruah , Theory of Fuzzy sets Beliefs and Realities ,IJEIC ,Vol. 2 ,Issue 2,1-22,2011