

## Control of Lorenz-Stenflo Chaotic System via Takagi-Sugeno Fuzzy Model Based on Linear Matrix Inequality

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### Abstract

*This paper investigates the chaos control problem for Lorenz-Stenflo chaotic system with uncertain parameters. Based on the interval matrix theory, the Takagi-Sugeno(T-S) fuzzy model is applied to the Lorenz-Stenflo chaotic system with uncertain parameters. Based on the Lyapunov stability theorem, the sufficient stability condition for the Lorenz-Stenflo chaotic system is presented as a set of linear matrix inequality (LMI) for the first time and the strict mathematical norms of LMI are given. Then the controller feedback gain can be obtained by solving a set of linear matrix inequality. Finally, simulation results for the Lorenz-Stenflo chaotic system are provided to illustrate the effectiveness of the proposed scheme.*

**Keywords:** *Lorenz-Stenflo chaotic system; uncertain parameters; Takagi-Sugeno fuzzy model; linear matrix inequality*

### 1. Introduction

Chaos as a very interesting nonlinear phenomenon has been intensively investigated in many fields of science and technology over the last decades, such as power system[1], secure communication[2], biology[3], time series[4], robot[5], mathematics [6], and so on. Therefore, chaos has attracted many researchers in all scientific fields[7].

As a research on chaos, Lorenz-Stenflo system has aroused increasing interests for many researchers. For example, some nonlinear dynamical behaviors of the Lorenz-Stenflo system have also been investigated[8]. Yang et al. [9] investigated the symplectic synchronization of Lorenz-Stenflo System via adaptive control. Chen *et al.*, [10] adopted single-variable substitution control for global synchronization criteria for two Lorenz-Stenflo systems.

Nowadays, different techniques and methods have been proposed to achieve chaos control. Chaos control has great significance in the application of chaos. Since chaos is very sensitive to its initial condition, chaos control was once believed to be impossible. However, the OGY method [11], developed in the 1990s, completely changed this situation. It has been an attractive research area, and many researchers have made lots of significant contributions to it. For example, based on the tracking control and the stability theory of nonlinear fractional-order systems, a new type of fractional-order chaotic synchronization which has multidrive systems and one response system is presented by Zhou *et al.*, [12]. Gao et al. [13] proposed the universal fuzzy model and universal fuzzy controller for stochastic non-affine nonlinear systems. Zhou *et al.*, [14] achieved chaos synchronization for the fractional-order Lorenz chaotic system via fractional-order derivative. Yang *et al.*, [15] proposed a control method of chaos in Lorenz system, whereby a sliding surface is assigned such that a sliding mode motion occurs when the proposed control law is applied. Since then, the sliding mode control for chaos has been a hot topic. For example, Wang *et al.*, [16] investigated synchronization of uncertain fractional-order chaotic systems with disturbance via sliding

mode control. Chen *et al.*, [17]. Proposed a sliding mode controller which realized the control of a class of fractional-order chaotic systems.

However, most of the above methods are concerned about the control of systems with certain parameters. For the uncertain parameters, many researchers adopt sliding mode control approach. However, it should be noted that the chattering phenomenon exists in sliding mode control[18]. Moreover, it will take too much time to control the state trajectories to the fixed point or the periodic orbit. These problems limit the sliding mode control method for systems with uncertain parameters.

Motivated by the above discussion, there are three advantages of our approach. First, the more practical sufficient stability condition for the T-S fuzzy control of the Lorenz-Stenflo system is proposed for the first time. Second, the controller is developed to stabilize the Lorenz-Stenflo chaotic system even if the system with five uncertain parameters. Finally, numerical simulations show that the proposed method can easily and quickly eliminate chaos as well as control the system to the equilibrium point.

The paper is presented as follows: In Section 2, the general description and nonlinear dynamics analysis of chaotic Lorenz-Stenflo system are presented. In Section 3, the T-S fuzzy model and the stability analysis of the closed-loop system are described. The controller design scheme based on PDC is also included in this section. In Section 4, numerical simulations results are shown. The brief comments and conclusions are drawn in Section 5.

## 2. System Description and Nonlinear Dynamics Analysis

### 2.1. Lorenz-Stenflo System

The Lorenz-Stenflo system is described as follows:

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + rx_4, \\ \dot{x}_2 = dx_1 - x_2 - x_1x_3, \\ \dot{x}_3 = -bx_3 + x_1x_2, \\ \dot{x}_4 = -x_4 - cx_1. \end{cases} \quad (1)$$

where  $a = 11, b = 2.9, c = 5, d = 23, r = 1.9$ . When the initial condition is selected as  $[x_1 \ x_2 \ x_3 \ x_4] = [0.005 \ 0.1 \ 0.2 \ 0.2]$ , the chaotic behavior of system (1) is shown in Figure 1.

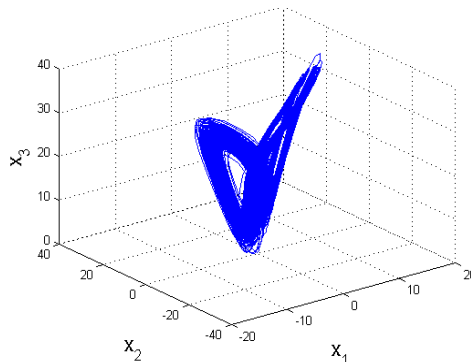


Figure 1. State Trajectories of Lorenz-Stenflo System

## 2.2. The Bifurcation Map of the System

The bifurcation map is used to analyze the dynamic characteristics of nonlinear system when the system parameter varies. Bifurcation is the main route to chaos from the stable state. The bifurcation diagram is shown in Figure 2 with  $0.6 < c < 1.6$ .

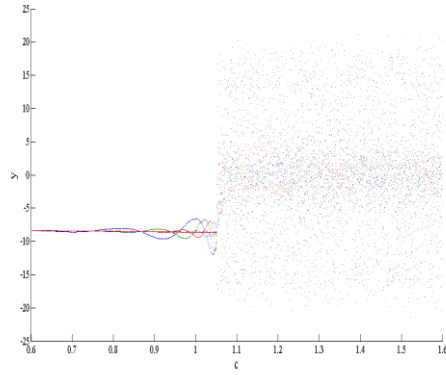


Figure 2. Bifurcation Map of Lorenz-Stenflo System

## 2.3. Poincare Map

Poincare map is a classic technology of dynamical system analysis, and the occurrence of chaos can be determined by the distribution of cut points on the Poincare section: when the dense point on the Poincare section is flaky and structural, the system is chaotic. The Poincare map of Lorenz-Stenflo system is shown in Figure 3.

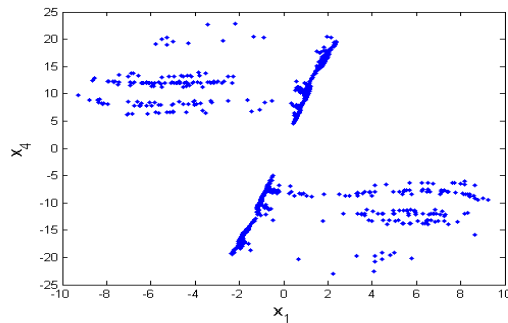
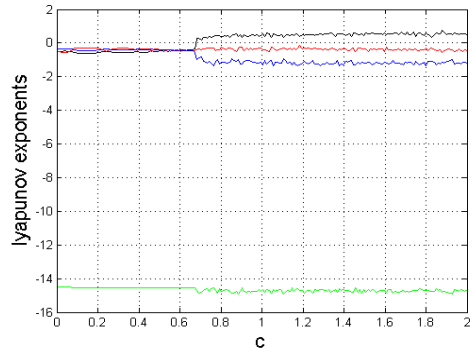


Figure 3. Poincare Map of Lorenz-Stenflo System

## 2.4. Lyapunov Exponent Spectrum

To make the chaos more intuitive, a diagram of the Lyapunov exponents varying with the parameter  $c$  is presented in Figure 4. One can observe that the largest Lyapunov exponent is above 0 when  $c = 0.7$ , which means that the system is chaotic.



**Figure 4. The Lyapunov Exponents of Lorenz-Stenflo System**

### 3. T-S Fuzzy Control Method Design

#### 3.1. Problem Formulation

Consider the uncertain parameters in system (1), which can be modeled as follows:

$$\begin{cases} \dot{x}_1 = \bar{a}(x_2 - x_1) + \bar{r}x_4, \\ \dot{x}_2 = \bar{d}x_1 - x_2 - x_1x_3, \\ \dot{x}_3 = -\bar{b}x_3 + x_1x_2, \\ \dot{x}_4 = -x_4 - \bar{c}x_1. \end{cases} \quad (2)$$

where  $\bar{a} \in [a_1, a_2]$ ,  $\bar{b} \in [b_1, b_2]$ ,  $\bar{c} \in [c_1, c_2]$ ,  $\bar{d} \in [d_1, d_2]$ ,  $\bar{r} \in [r_1, r_2]$ .

The control objective is to design a stable controller to make the Lorenz-Stenflo system with uncertain parameter asymptotical stable to the equilibrium point. The following sections present the controller design method.

#### 3.2. T-S Fuzzy Model and the Parallel Distributed Compensation (PDC) Controller

The T-S fuzzy model [19] is described by fuzzy IF-THEN rules in which each rule locally represents a linear realization of the system over a certain region of the state space. The overall system is then an aggregation of these local linear system models. Suppose T-S fuzzy model is given in the following form:

Rule  $R^i$ : IF  $z_1(t)$  is  $M_{i1}$  and  $\dots$  and  $z_n(t)$  is  $M_{in}$

THEN  $\dot{x}(t) = A_i x(t) + B_i u(t)$  ( $i = 1, 2, \dots, r$ )

where  $M_{ij}$  ( $j = 1, 2, \dots, n$ ) is the fuzzy set and  $r$  is the number of IF-THEN rules,  $x(t) \in R^n$  is the state vector,  $A_i \in R^{n \times n}$ ,  $z(t) = [z_1(t), z_2(t), \dots, z_n(t)]$  is the premise variables,  $u(t)$  is the control input.

By using the singleton fuzzifier, product inference and center-average defuzzifier, the output of the fuzzy system can be expressed as

$$\dot{x}(t) = \sum_{i=1}^r h_i(z(t)) A_i x(t) + \sum_{i=1}^r h_i(z(t)) B_i u(t) \quad (3)$$

where

$$\left\{ \begin{array}{l} h_i(z(t)) = \frac{\prod_{j=1}^n M_{ij}(z_j(t))}{\sum_{i=1}^r \prod_{j=1}^n M_{ij}(z_j(t))} \geq 0, \\ \sum_{i=1}^r h_i(z(t)) = 1 \end{array} \right. \quad (4)$$

$M_{ij}(z(t))$  is the grade of membership of  $z_j(t)$  in  $M_{ij}$ . Here,  $h_i(z(t))$  can be regarded as the normalized weights of the IF-THEN rules.

The new fuzzy controller  $u(t)$  is designed on the basis of parallel distributed compensation (PDC) and represented as follows [20]:

Rule  $R^i$ : IF  $z_1(t)$  is  $M_{i1}$  and  $\dots$  and  $z_n(t)$  is  $M_{in}$   
THEN  $u(t) = K_i x(t)$  ( $i = 1, 2, \dots, r$ )

The inferred output of the PDC controller is expressed in the following form

$$u(t) = \sum_{j=1}^r h_j(z(t)) K_j x(t) \quad (5)$$

where  $K_i$  represents the feedback gain.

By substituting (5) into (3), we have

$$\dot{x}(t) = \sum_{i=1}^r h_i(z(t)) A_i x(t) + \sum_{i=1}^r h_i(z(t)) B_i \sum_{j=1}^r h_j(z(t)) K_j x(t) \quad (6)$$

In order to simplify Equ.(6), make the following treatment

$$\sum_{i=1}^r h_i A_i = h_1 A_1 + h_2 A_2 + \dots + h_r A_r \quad (7)$$

Considering  $\sum_{i=1}^r h_i(z(t)) = 1$  in (4), Equ.(7) can be rewritten as

$$\begin{aligned} & \sum_{i=1}^r h_i A_i \\ &= h_1(h_1 + h_2 + \dots + h_r) A_1 + \dots + h_r(h_1 + h_2 + \dots + h_r) A_r \\ &= (h_1^2 + h_1 h_2 + \dots + h_1 h_r) A_1 + \dots + (h_r h_1 + h_r h_2 + \dots + h_r^2) A_r \\ &= \sum_{i=1}^r \sum_{j=1}^r h_i h_j A_i \end{aligned} \quad (8)$$

For  $\sum_{i=1}^r h_i B_i \sum_{j=1}^r h_j K_j$  in (6), one gets

$$\begin{aligned}
 & \sum_{i=1}^r h_i B_i \sum_{j=1}^r h_j K_j \\
 &= (h_1 B_1 + h_2 B_2 + \dots + h_r B_r)(h_1 K_1 + h_2 K_2 + \dots + h_r K_r) \\
 &= h_1 B_1 (h_1 K_1 + \dots + h_r K_r) + \dots + h_r B_r (h_1 K_1 + h_2 K_2 + \dots + h_r K_r) \\
 &= h_1^2 B_1 K_1 + h_2^2 B_2 K_2 + \dots + h_i^2 B_i K_i + \dots + h_r^2 B_r K_r + h_1 (h_2 K_2 + h_3 K_3 + \dots + h_r K_r) B_1 \\
 &\quad + h_2 (h_3 K_3 + \dots + h_r K_r) B_2 + \dots + h_i (h_{i+1} K_{i+1} + \dots + h_r K_r) B_i + \dots + h_{r-1} h_r K_r B_{r-1} \\
 &\quad + h_2 h_1 K_1 B_2 + h_3 (h_2 K_2 + h_1 K_1) B_3 + \dots + h_j (h_{j-1} K_{j-1} + h_{j-2} K_{j-2} + \dots + h_1 K_1) B_j \\
 &\quad + \dots + h_r (h_{r-1} K_{r-1} + h_{r-2} K_{r-2} + \dots + h_1 K_1) B_r \\
 &= \sum_{i=1}^r h_i^2 B_i K_i + \sum_{i < j} h_i h_j K_j B_i + \sum_{i > j} h_i h_j K_j B_i \\
 &= \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i K_j
 \end{aligned} \tag{9}$$

By substituting (8) and (9) to (6), one gets

$$\dot{x}(t) = \sum_{i=1}^r \sum_{j=1}^r h_i(z(t)) h_j(z(t)) (A_i + B_i K_j) x(t) \tag{10}$$

### 3.3. The Interval Matrix theory

In order to get the coefficient matrix of the Lorenz-Stenflo chaotic system with uncertain parameters, the following lemma is introduced:

**Lemma 1** [21] The following linear system:

$$\dot{x}(t) = \tilde{A} x(t) \tag{11}$$

where  $x(t) \in R^n$ . The elements of matrix  $\tilde{A}$  cannot be determined completely, but they belong to some certain interval. Matrix  $\tilde{A}$  is called the interval matrix, and the system (11) is called the interval system. In general, the interval matrix can be described as

$$\tilde{A} \in N[P, Q] = \left\{ A \in R^{n \times n} \mid p_{ij} \leq a_{ij} \leq q_{ij}, i, j = 1, \dots, n \right\}$$

where  $P$  is the lower bounds of matrix  $\tilde{A}$  while  $Q$  is the upper bounds.

The matrix  $\tilde{A}$  can be equally written as

$$\tilde{A} = A_{0i} + E \sum_i F \tag{12}$$

where

$$A_{0i} = \frac{1}{2} (P_i + Q_i),$$

$$\sum_i \in \Sigma^* = \left\{ \Sigma \in R^{n \times n} \mid \Sigma = \text{diag}(\varepsilon_{11}, \dots, \varepsilon_{1n}, \dots, \varepsilon_{n1}, \dots, \varepsilon_{nn}), |\varepsilon_{ij}| \leq 1, i, j = 1, \dots, n \right\},$$

$$E = (\sqrt{h_{11}} e_1, \dots, \sqrt{h_{1n}} e_1, \dots, \sqrt{h_{n1}} e_n, \dots, \sqrt{h_{nn}} e_n),$$

$$F = (\sqrt{h_{11}} e_1, \dots, \sqrt{h_{1n}} e_n, \dots, \sqrt{h_{n1}} e_1, \dots, \sqrt{h_{nn}} e_n)^T,$$

$H = (h_{ij})_{n \times n} = H_i = \frac{1}{2} (Q_i - P_i)$ ,  $e_i (i = 1, \dots, n)$  is the  $i$  th column of  $n \times n$  identity matrix.

Note that for the arbitrary  $i$  and  $\Sigma_i \in \Sigma^*$ , one has

- 1)  $\Sigma_i \Sigma_i^T = \Sigma_i^T \Sigma_i \leq I$  ( $I$  is  $n \times n$  identity matrix);
- 2)  $EE^T = \text{diag} \left\{ \sum_{j=1}^n h_{1j}, \dots, \sum_{j=1}^n h_{nj} \right\}$ ;
- 3)  $F^T F = \text{diag} \left\{ \sum_{i=1}^n h_{i1}, \dots, \sum_{i=1}^n h_{in} \right\}$ .

### 3.4. T-S Fuzzy Model for Lorenz-Stenflo Chaotic System

To utilize the LMI-based fuzzy system design techniques, we start with representing chaotic systems using T-S fuzzy models in 3.2. Before putting this model into the Lorenz-Stenflo system, make the following assumptions:

**Assumption 1:** According the boundedness of the chaos system, the scope of the system state trajectory can be defined as

$$\Omega \equiv \left\{ x(t) \in R \mid \|x(t)\|^n \leq \delta \right\}, x_i(t) \in [-d, d], d > 0 (i = 1, 2).$$

**Assumption 2:** The feedback control is applied to each state of the system. With the T-S fuzzy model, the PDC method can be adopted to control the Lorenz-Stenflo system.

**Assumption 3:** The premise variables are independent of the control input of the model.

According to the assumptions above, together with the interval matrix theory, the T-S fuzzy model can be applied to describe the Lorenz-Stenflo chaotic system with uncertain parameters. Suppose that  $x_1(t) \in [-d_1, d_1]$ ,  $x_2(t) \in [-d_2, d_2]$ , where  $d_1 = 20$ ,  $d_2 = 25$ . Then the system is described in T-S fuzzy model as follows

$$R^1: \text{IF } x_1(t) \text{ is } M_1(x_1(t)), \text{ THEN } \dot{x}(t) = \tilde{A}_1 x(t) + B_1 u(t),$$

$$R^2: \text{IF } x_1(t) \text{ is } M_2(x_1(t)), \text{ THEN } \dot{x}(t) = \tilde{A}_2 x(t) + B_2 u(t),$$

$$R^3: \text{IF } x_2(t) \text{ is } M_3(x_2(t)), \text{ THEN } \dot{x}(t) = \tilde{A}_3 x(t) + B_3 u(t),$$

$$R^4: \text{IF } x_2(t) \text{ is } M_4(x_2(t)), \text{ THEN } \dot{x}(t) = \tilde{A}_4 x(t) + B_4 u(t).$$

where  $x = [x_1(t), x_2(t), x_3(t), x_4(t)]^T$  and  $u(t) = [u_1(t), u_2(t), u_3(t), u_4(t)]^T$ ,

$$\tilde{A}_1 = \begin{bmatrix} \tilde{a} & \tilde{a} & 0 & \tilde{r} \\ \tilde{d} & -1 & -d_1 & 0 \\ 0 & d_1 & -\tilde{b} & 0 \\ -\tilde{c} & 0 & 0 & -1 \end{bmatrix}, \tilde{A}_2 = \begin{bmatrix} \tilde{a} & \tilde{a} & 0 & \tilde{r} \\ \tilde{d} & -1 & d_1 & 0 \\ 0 & -d_1 & -\tilde{b} & 0 \\ -\tilde{c} & 0 & 0 & -1 \end{bmatrix}, \tilde{A}_3 = \begin{bmatrix} \tilde{a} & \tilde{a} & 0 & \tilde{r} \\ \tilde{d} & -1 & 0 & 0 \\ d_2 & 0 & -\tilde{b} & 0 \\ -\tilde{c} & 0 & 0 & -1 \end{bmatrix}, \tilde{A}_4 = \begin{bmatrix} \tilde{a} & \tilde{a} & 0 & \tilde{r} \\ \tilde{d} & -1 & 0 & 0 \\ -d_2 & 0 & -\tilde{b} & 0 \\ -\tilde{c} & 0 & 0 & -1 \end{bmatrix}.$$

$$B_1 = B_2 = B_3 = B_4 = I_{4 \times 4} \text{ (} I \text{ is the identity matrix).}$$

The membership function of the fuzzy sets can be selected as

$$M_1(x_1(t)) = \frac{1}{2} \left( 1 + \frac{x_1(t)}{d_1} \right), M_2(x_1(t)) = \frac{1}{2} \left( 1 - \frac{x_1(t)}{d_1} \right),$$

$$M_3(x_2(t)) = \frac{1}{2} \left(1 + \frac{x_2(t)}{d_2}\right), \quad M_4(x_2(t)) = \frac{1}{2} \left(1 - \frac{x_2(t)}{d_2}\right).$$

Therefore, the Lorenz-Stenflo system based on T-S fuzzy model can be represented as

$$\dot{x}(t) = \sum_{i=1}^4 h_i(z(t))(A_i x(t) + B_i u(t)) \quad (13)$$

### 3.5. T-S Fuzzy Model-based Controller

According to the parallel distributed compensation (PDC) techniques in 3.2, the controller is described in T-S fuzzy model as follows

$$R^1: \text{IF } x_1(t) \text{ is } M_1(x_1(t)), \text{ THEN } u(t) = K_1 x(t),$$

$$R^2: \text{IF } x_1(t) \text{ is } M_2(x_1(t)), \text{ THEN } u(t) = K_2 x(t),$$

$$R^3: \text{IF } x_2(t) \text{ is } M_3(x_2(t)), \text{ THEN } u(t) = K_3 x(t),$$

$$R^4: \text{IF } x_2(t) \text{ is } M_4(x_2(t)), \text{ THEN } u(t) = K_4 x(t).$$

The total input of the controller is obtained as

$$u(t) = \sum_{i=1}^4 h_i(z(t)) K_i x(t) \quad (14)$$

Substitute (14) into (13), one can get the closed-loop Lorenz-Stenflo system

$$\dot{x}(t) = \sum_{i=1}^4 \sum_{j=1}^4 h_i(z(t)) h_j(z(t)) (A_i + B_i K_j) x(t) \quad (15)$$

From the interval matrix theory in 3.3, Equ.(15) can be rewritten as

$$\dot{x}(t) = \sum_{i=1}^4 \sum_{j=1}^4 h_i(z(t)) h_j(z(t)) (A_{0i} + E \sum_i F + B_i K_j) x(t) \quad (16)$$

Here, the definition of  $A_{0i} (i = 1, 2, 3, 4)$ ,  $E$ ,  $F$ ,  $\sum_i$  can refer to Lemma 1.

To design the stable controller (14) for system (15), the following lemma and theorem are given.

**Lemma 2** If  $\xi$  and  $\theta$  are any proper dimensional column vector, for the arbitrary constant  $\eta > 0$ , one has

$$2\xi^T \Sigma \theta \leq \eta \xi^T \xi + \eta^{-1} \theta^T \theta, \quad \forall \Sigma \in \Sigma^* \quad (17)$$

**Theorem 1** For all the subsystem of the closed-loop system(15), if there exists a positive definite matrix  $P$  as well as a positive constant  $\eta$ , and select the controller gain matrix  $K_i (i = 1, 2, 3, 4)$  which satisfies the following inequality, the T-S fuzzy model dynamical equation(15) is globally asymptotically stable.

$$G_{ii}^T P + P G_{ii} + \eta P E E^T P + \eta^{-1} F^T F < 0 \quad (i, j = 1, 2, 3, 4) \quad (18)$$

$$G_{ij}^T P + P G_{ij} + \eta P E E^T P + \eta^{-1} F^T F < 0 \quad (i < j \leq 4) \quad (19)$$

Here,  $G_{ii} = A_{0i} + B_i K_i$ ,  $G_{ij} = \frac{(A_{0i} + B_i K_j) + (A_{0j} + B_j K_i)}{2}$

**Proof.** Consider  $V(x) = x^T P x$  ( $P$  is a positive definite matrix) as Lyapunov candidate function of the system (15). The time derivation along the solution of system (15) is



$$\begin{aligned}
 V(x) &= x^T P x + x^T P x \\
 &= \sum_{i=1}^4 \sum_{j=1}^4 h_i h_j \left[ (A_{0i} + E \Sigma F + B_i K_j) x \right]^T P x + x^T P \sum_{i=1}^4 \sum_{j=1}^4 h_i h_j (A_{0i} + E \Sigma F + B_i K_j) x \\
 &= \sum_{i=1}^4 \sum_{j=1}^4 h_i h_j x^T \left\{ \left[ (A_{0i} + B_i K_j)^T P + F^T \Sigma E^T P \right] + \left[ P (A_{0i} + B_i K_j) + P E \Sigma F \right] \right\} x \\
 &= \sum_{i=1}^4 h_i^2 x^T \left\{ \left[ (A_{0i} + B_i K_j)^T P + F^T \Sigma E^T P \right] + \left[ P (A_{0i} + B_i K_j) + P E \Sigma F \right] \right\} x + \\
 &\quad \sum_{i < j}^4 h_i h_j x^T \left[ (A_{0i} + B_i K_j)^T P + F^T \Sigma E^T P + P (A_{0i} + B_i K_j) + P E \Sigma F \right] x + \\
 &\quad \sum_{i > j}^4 h_i h_j x^T \left[ (A_{0j} + B_j K_i)^T P + P (A_{0j} + B_j K_i) \right] x \\
 &= \sum_{i=1}^4 h_i^2 x^T \left[ (A_{0i} + B_i K_i)^T P + P (A_{0i} + B_i K_i) \right] x + \sum_{i=1}^4 h_i^2 x^T (F^T \Sigma E^T P + P E \Sigma F) x + \\
 &\quad 2 \sum_{i < j}^4 h_i h_j x^T (F^T \Sigma E^T P + P E \Sigma F) x + \sum_{i < j}^4 h_i h_j x^T \left[ (A_{0i} + B_i K_j)^T P + P (A_{0i} + B_i K_j) \right] x + \\
 &\quad \sum_{i > j}^4 h_i h_j x^T \left[ (A_{0j} + B_j K_i)^T P + P (A_{0j} + B_j K_i) \right] x
 \end{aligned}$$

From (4), we can get  $\sum_{i=1}^4 h_i^2 + 2 \sum_{i < j}^4 h_i h_j = 1$ , and

$$\begin{aligned}
 &\sum_{i < j}^4 h_i h_j x^T \left[ (A_{0i} + B_i K_j)^T P + P (A_{0i} + B_i K_j) \right] x + \sum_{i > j}^4 h_i h_j x^T \left[ (A_{0j} + B_j K_i)^T P + P (A_{0j} + B_j K_i) \right] x \\
 &= \sum_{i < j}^4 h_i h_j x^T \left\{ \left[ (A_{0i} + B_i K_j) + (A_{0j} + B_j K_i) \right]^T P + P \left[ (A_{0i} + B_i K_j) + (A_{0j} + B_j K_i) \right] \right\} x \\
 &= 2 \sum_{i < j}^4 h_i h_j x^T \left\{ \left[ \frac{(A_{0i} + B_i K_j) + (A_{0j} + B_j K_i)}{2} \right]^T P + P \left[ \frac{(A_{0i} + B_i K_j) + (A_{0j} + B_j K_i)}{2} \right] \right\} x
 \end{aligned}$$

Select  $G_{ii} = A_{0i} + B_i K_i$ ,  $G_{ij} = \frac{(A_{0i} + B_i K_j) + (A_{0j} + B_j K_i)}{2}$ . One has

$$\begin{aligned}
 V(x) &= \sum_{i=1}^4 h_i^2 x^T (G_{ii}^T P + P G_{ii}) x + 2 \sum_{i < j}^4 h_i h_j x^T (G_{ij}^T P + P G_{ij}) x + \\
 &\quad x^T F^T \Sigma E^T P x + x^T P E \Sigma F x
 \end{aligned} \tag{20}$$

From Lemma 2, select  $\xi^T = x^T P E$ ,  $\theta = F x$ , one can get

$$2 x^T P E \Sigma F x \leq \eta x^T P E E^T P x + \eta^{-1} x^T F^T F x \tag{21}$$

Taking the transpose of both sides in (21), we obtain

$$2 x^T F^T \Sigma E^T P x \leq \eta x^T P E E^T P x + \eta^{-1} x^T F^T F x \tag{22}$$

According to (21) and (22), we get

$$x^T F^T \Sigma E^T P x + x^T P E \Sigma F x \leq \eta x^T P E E^T P x + \eta^{-1} x^T F^T F x \tag{23}$$

By substituting (23) into (20), one has

$$V(x) \leq \sum_{i=1}^4 h_i^2 x^T (G_{ii}^T P + P G_{ii}) x + 2 \sum_{i < j}^4 h_i h_j x^T (G_{ij}^T P + P G_{ij}) x + \eta x^T P E E^T P x + \eta^{-1} x^T F^T F x \quad (24)$$

Note that  $\sum_{i=1}^4 h_i^2 + 2 \sum_{i < j}^4 h_i h_j = 1$ , so (24) can be equally represented as

$$V(x) \leq \sum_{i=1}^4 h_i^2 x^T (G_{ii}^T P + P G_{ii} + \eta P E E^T P + \eta^{-1} F^T F) x + 2 \sum_{i < j}^4 h_i h_j x^T (G_{ij}^T P + P G_{ij} + \eta P E E^T P + \eta^{-1} F^T F) x$$

From (18) and (19) in the Theorem 1, it can be obtained that

$$V(x) = x^T P x + x^T P x < 0.$$

That is to say, the chaotic system (15) with the controller (14) is globally asymptotically stable. This completes the proof.

The theorem 1 gives a sufficient condition for the stability of the closed-loop Lorenz-Stenflo system with uncertain parameters. And (18) and (19) can be transformed into the problem of solving linear matrix inequalities (LMIs).

From (18), one has

$$A_i^T P + K_i^T B_i P + P A_i + P B_i K_i + \eta P E E^T P + \eta^{-1} F^T F < 0 \quad (25)$$

Multiplying  $P^{-1}$  both in the left and right in (25), we obtain

$$P^{-1} A_i^T + P^{-1} K_i^T B_i + A_i P^{-1} + B_i K_i P^{-1} + \eta E E^T + \eta^{-1} P^{-1} F^T F P^{-1} < 0 \quad (26)$$

Select  $Q = P^{-1}$ ,  $M_i = K_i P^{-1}$  in (26), one gets

$$Q A_i^T + A_i Q + M_i^T B_i^T + B_i M_i + \eta E E^T + \eta^{-1} Q F^T F Q < 0 \quad (27)$$

The gain of the controller is  $K_i = M_i Q^{-1}$ .

From (19), one has

$$A_i^T P + K_j^T B_j^T P + A_j^T P + K_i^T B_j^T P + P A_i + P B_i K_j + P A_j + P B_j K_i + \eta P E E^T P + \eta^{-1} F^T F < 0 \quad (28)$$

Multiplying  $P^{-1}$  both in the left and right in (28), we obtain

$$P^{-1} A_i^T + P^{-1} K_j^T B_j^T + P^{-1} A_j^T + P^{-1} K_i^T B_j^T + A_i P^{-1} + B_i K_j P^{-1} + A_j P^{-1} + B_j K_i P^{-1} + \eta E E^T + \eta^{-1} P^{-1} F^T F P^{-1} < 0 \quad (29)$$

Select  $Q = P^{-1}$ ,  $M_i = K_i P^{-1}$ ,  $M_j = K_j P^{-1}$  in (29), one gets

$$Q A_i^T + A_i Q + M_i^T B_j^T + B_j M_i + Q A_j^T + A_j Q + M_j^T B_i^T + B_i M_j + \eta E E^T + \eta^{-1} Q F^T F Q < 0 \quad (30)$$

The gain of the controller is  $K_i = M_i Q^{-1}$ .

According to Schur's theorem[21], the standard form of the linear matrix inequality is described as follows:

From (27), this yields

$$\begin{bmatrix} Q A_i^T + M_i^T B_i^T + A_i Q + B_i M_i + \eta E E^T & Q F^T \\ & \eta^{-1} I \\ F Q & \end{bmatrix} < 0 \quad (31)$$

From (30), this yields

$$\begin{bmatrix} QA_i^T + A_i^T Q + M_i^T B_j^T + B_j M_i + QA_j^T + A_j^T Q + M_j^T B_i^T + B_i M_j + \eta EE^T & QF^T \\ & FQ \\ & & -\eta^{-1}I \\ & & & Q > 0 \end{bmatrix} < 0 \quad (32)$$

$$Q > 0 \quad (33)$$

where  $I$  is  $4 \times 4$  identity matrix.

For a given  $\eta > 0$ , the positive definite matrix  $P$  and the controller gain  $K_i$  can be obtained via Matlab's LMI toolbox by solving the inequality (31), (32) and (33).

#### 4. Simulation Results

Considering the uncertain parameters in Lorenz-Stenflo chaotic system, the parameters can be chosen as  $\tilde{a} = 11 + 0.05 \sin(t)$ ,  $\tilde{b} = 2.9 + 0.1 \cos(t)$ ,  $\tilde{c} = 5 + 0.005 \sin(t)$ ,  $\tilde{d} = 23 + 0.1 \cos(t)$ ,  $\tilde{r} = 1.9 + 0.1 \sin(t)$ , i.e.  $\tilde{a} \in [10.95, 11.05]$ ,  $\tilde{b} \in [2.8, 3.0]$ ,  $\tilde{c} \in [4.995, 5.005]$ ,  $\tilde{d} \in [22.9, 23.1]$ ,  $\tilde{r} \in [1.8, 2.0]$ . Select  $d_1 = 20, d_2 = 25$ . Take control of the Lorenz-Stenflo chaotic system (15) with uncertain parameters, where the values of  $A_{0i} (i = 1, 2, 3, 4)$  and  $E, F$  can be calculated through Lemma 1.  $B_1 = B_2 = B_3 = B_4 = I_{4 \times 4}$ .

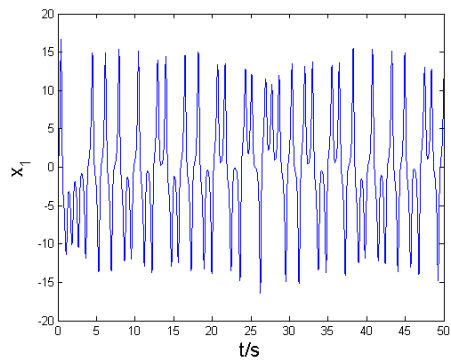
According to the Theorem 1, select  $\eta = 1000$ , through Matlab's LMI toolbox, we can obtain

$$P = \text{diag}(0.0040, 0.0020, 0.0029, 0.0029),$$

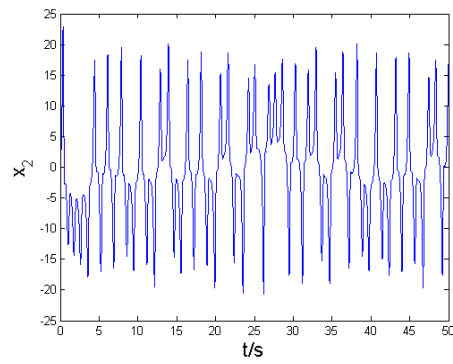
$$K_1 = \begin{bmatrix} 8.6056 & -44.4386 & -35.4146 & 28.2826 \\ 44.5208 & -0.0874 & -7.5381 & -2.7674 \\ 48.1115 & -1.4728 & 1.2839 & 36.3687 \\ -36.0037 & 1.8619 & -36.3687 & -0.4769 \end{bmatrix}, K_2 = \begin{bmatrix} 8.6056 & -18.9165 & 26.3964 & -0.9068 \\ -7.0146 & -0.0874 & 13.3957 & -3.1577 \\ -35.8601 & -2.4681 & 1.2839 & 21.4724 \\ 3.6508 & 2.1245 & -21.4724 & -0.4769 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 8.6056 & -8.3569 & -14.5569 & -0.3917 \\ -28.3372 & -0.0874 & 15.2190 & 12.2641 \\ -5.2242 & -10.2391 & 1.2839 & -9.4830 \\ 2.9509 & -8.2511 & 9.4830 & -0.4769 \end{bmatrix}, K_4 = \begin{bmatrix} 8.6056 & -11.0177 & 9.2403 & 0.9775 \\ -22.9642 & -0.0874 & -0.2248 & 0.2480 \\ 12.4469 & 0.1513 & 1.2839 & 0.0319 \\ 1.0909 & -0.1669 & -0.0319 & -0.4769 \end{bmatrix}$$

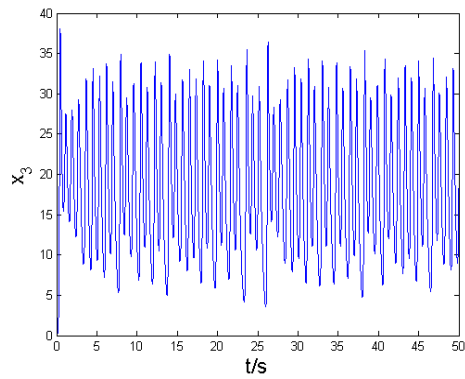
By substituting the value of  $K_i (i = 1, 2, 3, 4)$  into (14), the desired controller (14) can be got. Figure 5 illustrates the state trajectory of  $x_1, x_2, x_3, x_4$  of Lorenz-Stenflo system without controller. From Figure 5, we learn that the whole system is unstable before taking control of it. Figure 6 shows the state trajectory of closed-loop Lorenz-Stenflo chaotic system with the controller (14). From Figure 6, we can clearly see that after the controller was applied, the states of system (15) globally converge to the equilibrium point immediately which implies the effectiveness of the designed controller. The controller feedback gain for Lorenz-Stenflo chaotic system can be obtained by solving a set of LMI which is more practical. And the controller can stabilize the Lorenz-Stenflo chaotic system even if the system with several uncertain parameters which shows the advantage of robustness.



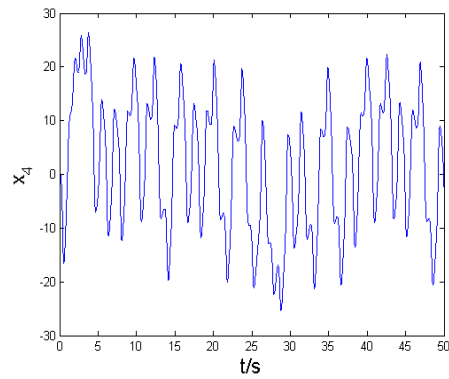
(a) State Trajectory of  $x_1$



(b) State Trajectory of  $x_2$

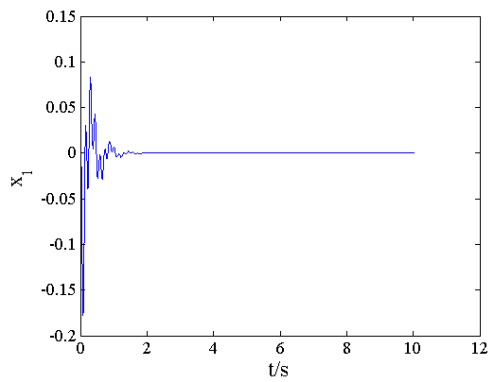


(c) State Trajectory of  $x_3$

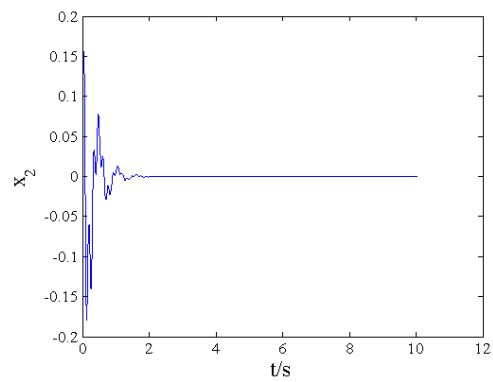


(d) State Trajectory of  $x_4$

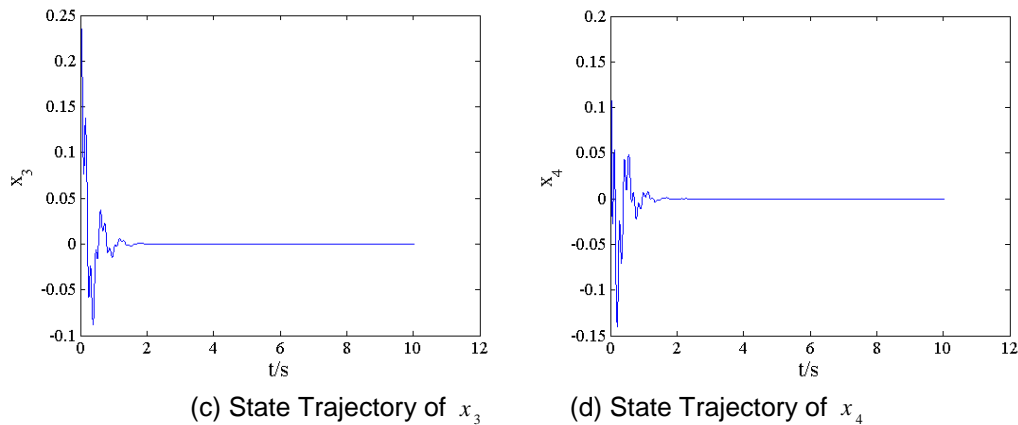
**Figure 5. State Trajectories of Lorenz-Stenflo System without Controller in the Presence of Uncertainty**



(a) State Trajectory of  $x_1$



(b) State Trajectory of  $x_2$



**Figure 6. State Trajectories of Lorenz-Stenflo System with Controller in the Presence of Uncertainty**

## 5. Conclusion

In this paper, T-S fuzzy mode method was applied to control the Lorenz-Stenflo chaotic system. The complex dynamics characters of the system were presented, including the phase trajectory, the Lyapunov exponents, the Poincare map and the bifurcation map. According to the Lyapunov stability theorem together with the parallel distributed compensation techniques, a fuzzy mode control law has been designed to control chaos in the system. The parameters of the controller could be selected by solving a linear matrix inequality. Based on the fuzzy mode control method, the states of the system with uncertainties have been stabilized. Numerical simulations were given to verify the effectiveness and the strong robustness of the proposed scheme.

## Acknowledgements

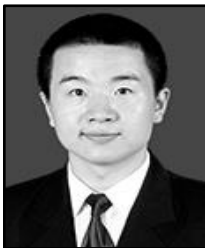
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