Parameterization of Some Surfaces of Revolution Through Curvature-Varying Curves: A Computational Analysis

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Abstract

This paper describes a computational analysis of various parameterizations of a surface of revolution. The idea is to generate the parameterization of the surface by using different (not necessarily planar) curves with varying curvature. The approach to the practical case of obtaining the parameterization of a torus through different planar curves other than circles, such as planar lemniscates. The analysis is also extended to the interesting case of non-planar curves. Our work relies on the intensive use of Mathematica, a powerful computational system very well suited for symbolic calculations while also providing valuable numerical and graphical options, a user-friendly graphical interface and a compact and efficient programming language. All calculations in this paper have been carried out with Mathematica v8.0.

1. Introduction

The torus is one of the most intriguing, useful and exciting surfaces of revolution in geometry, with outstanding applications in several scientific fields, such as mathematics, computer graphics, electromagnetism, nuclear science, and many others [1, 2, 9, 11, 12]. Intuitively, the torus is a three-dimensional surface resembling the shape of a doughnut; the tradition of cooking food with that shape can be traced back to the ancient times of the Roman Empire and even earlier. In electro- magnetism, these surfaces are used to create a magnet by using a coil wound a certain number of turns on a torus to generate a magnetic field by means of an electric current through the coil. The torii are also used in nuclear physics in devices such as the tokamak, to achieve a stable plasma equilibrium by using magnetic field lines that move around the torus in a helical shape. In video games industry, toroidal scenarios are often applied to reuse the bots and even the main characters of the game as if they go around the world, so that the player feels immersed in a virtual sky- dome. The torus also appears in many other practical uses, such as the O-rings traditionally found in mechanical engineering to create a seal at the interface of machines such as rotating pump shafts and hydraulic cylinder pistons, the inner tubes typically found in the tires of bicycles, motorcycles, buses, trucks and tractors, or simply the lifebuoys used in swimming pools in the past and still commonly found in many ships nowadays. Clearly, the range of applications of the torus goes far beyond the apparent simplicity of its shape. This justifies our interest to explore the parameterization of a torus from a computational perspective, the actual motivation of the present contribution.

Mathematically speaking, a torus T is simply a surface of revolution generated by revolving a circle C in the three-dimensional space R3 about an axis L coplanar with the circle. Depending on the relationship between the axis of revolution L and the surface T, different names are usually considered: if L does not touch the circle C, T has a ring shape and is called a ring torus or simply torus; if, however, the axis L is tangent to the circle C, the resulting surface T is called a horn torus; finally, when the axis L is a chord of the circle C, T is called a spindle torus. A degenerate case appears when the axis L is a diameter of the circle C, which simply generates the surface of a sphere (see [7, 8, 10] for further information about the mathematical basis of this field).

A torus can readily be defined parametrically: it is just a matter of considering a parameterization of circle C with respect to a reference system in R3 and then to apply a rotation matrix to obtain such a parameterization [13]. In this paper, we are interested to go further and analyze the following related questions:

1. Are there planar curves (other than circles) that can generate a torus as a surface of revolution?

2. Are there non-planar curves that can generate a torus as a surface of revolution?

3. If 1 and 2 hold, how do the toroidal curves obtained by applying the resulting parameterization onto linear curves look like?

4. If such toroidal curves are projected onto the coordinate planes, is it possible to find some kind of symmetry in those projected curves?

1.1. Aims and Structure of the Paper

In this work we try to answer these questions by following a computational approach. Our work relies on the intensive use of Mathematica, a powerful computational system very well suited for symbolic calculations [14] while also providing valuable numerical and graphical options, a user- friendly graphical interface and a compact and efficient programming language.

The structure of this paper is as follows: Section 2 provides some basic mathematical back-ground. The core of the paper is in Section 3, where some parameterizations of the torus through curvature-varying curves are obtained by means of symbolic computer manipulations. The section explores the cases of planar and non-planar curves. Answers to the questions posed above are provided. The paper closes with the main conclusions of our work and some further remarks.

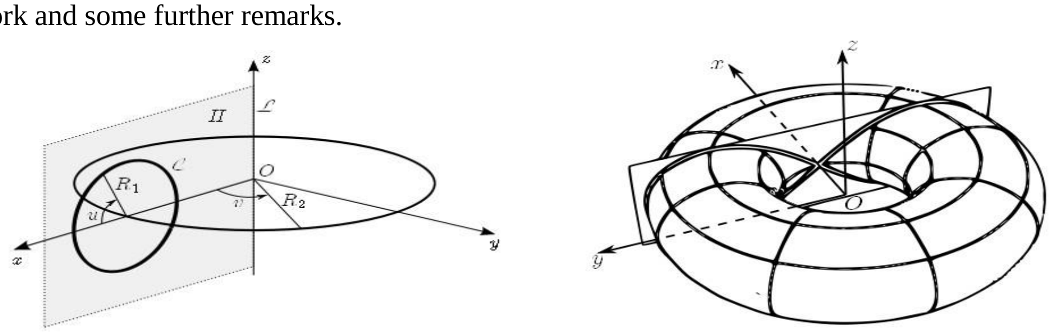

Figure 1. (Left) Graphical Scheme for the Parameterization of the Torus; (Right) Lemniscate Curve on the Torus

2. Basic Mathematical Background

Let us suppose that we are given a circle C and a line L coplanar with the circle but external to it. Without loss of generality, we can assume that L is the vertical axis z and the circle C lies on the vertical plane xz, represented onwards as Π, R1 > 0 is the distance from L to the center of C, and R2 (with 0 < R2 < R1) is the radius of C, according to Figure 1(left). Then, the torus T can be parameterized as:

$$\mathbf{T} = S(u,v) = \left[\begin{pmatrix} \cos(v) & -\sin(v) & 0 \\ \sin(v) & \cos(v) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} R_1 + R_2\cos(u) \\ 0 \\ R_2\sin(u) \end{pmatrix} \right]^T \quad (1)$$

or, equivalently:

$$\mathrm{T} = S(u,v) = (\cos(v)(R_1 + R_2\cos(u)), \sin(v)(R_1 + R_2\cos(u)), R_2\sin(u)) \quad (2)$$

where u and v are the parameters of the surface, both valued on the interval [0, 2π].

The implicit equation in Cartesian coordinates is given by:

$$\left(R_1 - \sqrt{x^2+y^2}\right)^2 + z^2 = R_2^2. \quad (3)$$

By algebraic elimination of the square root in Eq. (3), we get the quartic equation:

$$\left(x^2+y^2+z^2+R_1^2-R_2^2\right)^2 = 4R_1^2\left(x^2+y^2\right). \quad (4)$$

Intersecting now Eq. (4) with the plane x = R2 we obtain the contour curve defined by:

$$\left(y^2+z^2+R_1^2\right)^2 = 4R_1^2\left(y^2+R_2^2\right), \quad (5)$$

whose graphical representation is given by the lemniscate on the plane x = R2 displayed in Figure 1(right). This shows that it is possible to find lemniscate curves on a torus.

Our goal now is to obtain a parameterization of the torus in terms of some of those curves and then extend such a parameterization to other types of curves, as described in next section.

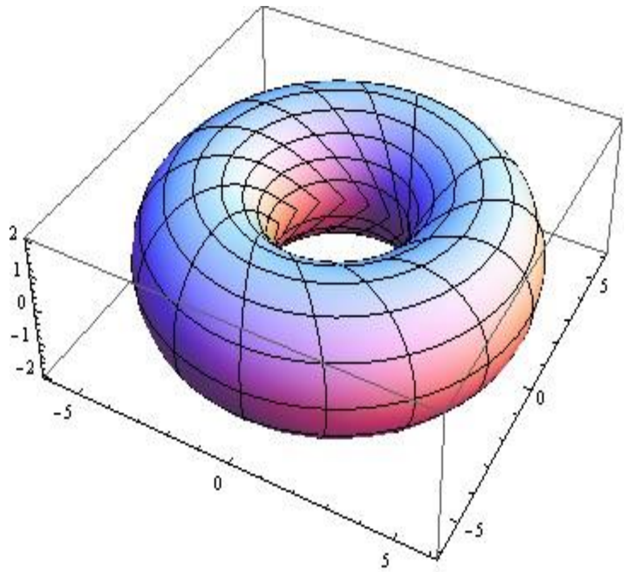

Figure 2. A Parameterization of a Torus through the Lemniscate Curve given by Eq. (6)

3. Parameterization of a Torus through Planar and Non-Planar Curvature-Varying Curves

In this section we derive different parameterizations of the torus through some curves of variable curvature. Similarly to the analysis carried out in [6], we begin our discussion with the case of planar curves. Then, the case of non-planar curves (*i.e.*, curves of nonzero torsion) is presented.

3.1. Case I: Planar Curves

3.1.1. Parameterization of a Torus through a Lemniscate Curve

The parametric equations of the lemniscate curve in Eq. (5) are given by:

$$\left(R_2, \frac{2\sqrt{2}\, R_2\, \cos(u)}{1+\sin^2(u)}, \frac{2\sqrt{2}\, R_2\, \cos(u) \sin(u)}{1+\sin^2(u)} \right) \tag{6}$$

Proceeding similarly to Eqs. (1)-(2), we can obtain a parameterization of the torus by applying a rotation to the lemniscate around the axis z. Making the calculations with Mathematica, we get:

$$\texttt{In[1]:=}\ \texttt{torus[u\_, v\_, R2\_]} := \left(\begin{pmatrix} \texttt{Cos}[v] & -\texttt{Sin}[v] & 0 \\ \texttt{Sin}[v] & \texttt{Cos}[v] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} R2 \\ \frac{2\sqrt{2}\, R2\ \texttt{Cos}[u]}{1+\texttt{Sin}[u]^2} \\ \frac{2\sqrt{2}\, R2\ \texttt{Cos}[u]\ \texttt{Sin}[u]}{1+\texttt{Sin}[u]^2} \end{pmatrix} \right)^{\mathrm{T}}$$

We can now compute this parameterization for any particular value of parameter R2, for instance,

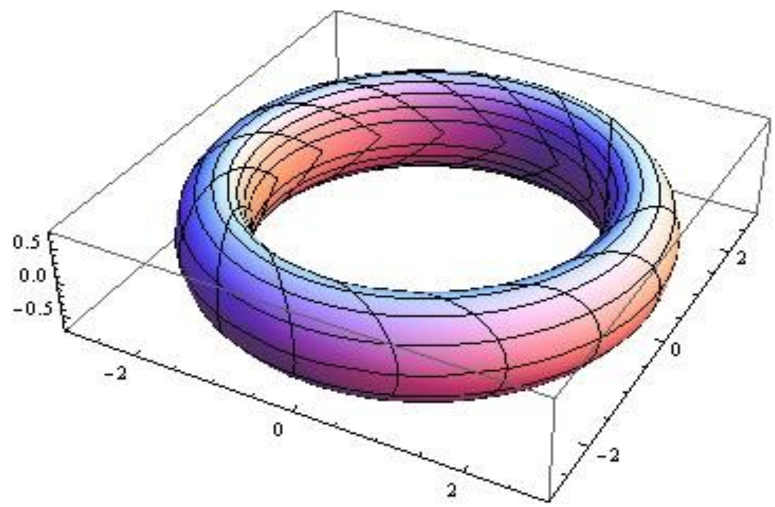

Figure 3. A Parameterization of a Torus through the Generalized Lemniscate Curve given by Eq. (7)

$R_2 = 2$ as:

In [2] := torus[u, v, 2]

Out [2]= $\left\{\left\{2\,\mathrm{Cos}[v] - \frac{4\sqrt{2}\,\mathrm{Cos}[u]\,\mathrm{Sin}[v]}{1+\mathrm{Sin}[u]^2}, \frac{4\sqrt{2}\,\mathrm{Cos}[u]\,\mathrm{Cos}[v]}{1+\mathrm{Sin}[u]^2} + 2\,\mathrm{Sin}[v], \frac{4\sqrt{2}\,\mathrm{Cos}[u]\,\mathrm{Sin}[u]}{1+\mathrm{Sin}[u]^2}\right\}\right\}$

In [3] := ParametricPlot3D [torus[u, v, 2], {u, −π/2, π/2}, {v, 0, 2π}]

Out [3] = *See Figure 2*

3.1.2. Parameterization of a Torus through a Generalized Lemniscate Curve

The parametric equations of the generalized lemniscate curve lying on the plane x = b are given by:

$$\left(b, \frac{a\cos(u)}{1+\sin^2(u)}, \frac{a\cos(u)\sin(u)}{1+\sin^2(u)}\right) \tag{7}$$

Proceeding similarly to the previous case, we get the following parameterization of the torus:

In [4] := torusgen[u_, v_, a_, b_] := $\left(\begin{pmatrix}\mathrm{Cos}[v] & -\mathrm{Sin}[v] & 0\\ \mathrm{Sin}[v] & \mathrm{Cos}[v] & 0\\ 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix} b \\ \frac{a\,\mathrm{Cos}[u]}{1+\mathrm{Sin}[u]^2} \\ \frac{a\,\mathrm{Cos}[u]\,\mathrm{Sin}[u]}{1+\mathrm{Sin}[u]^2}\end{pmatrix}\right)^{\mathrm{T}}$

For instance:

In[5]:= torusgen[u, v, 2, 2]

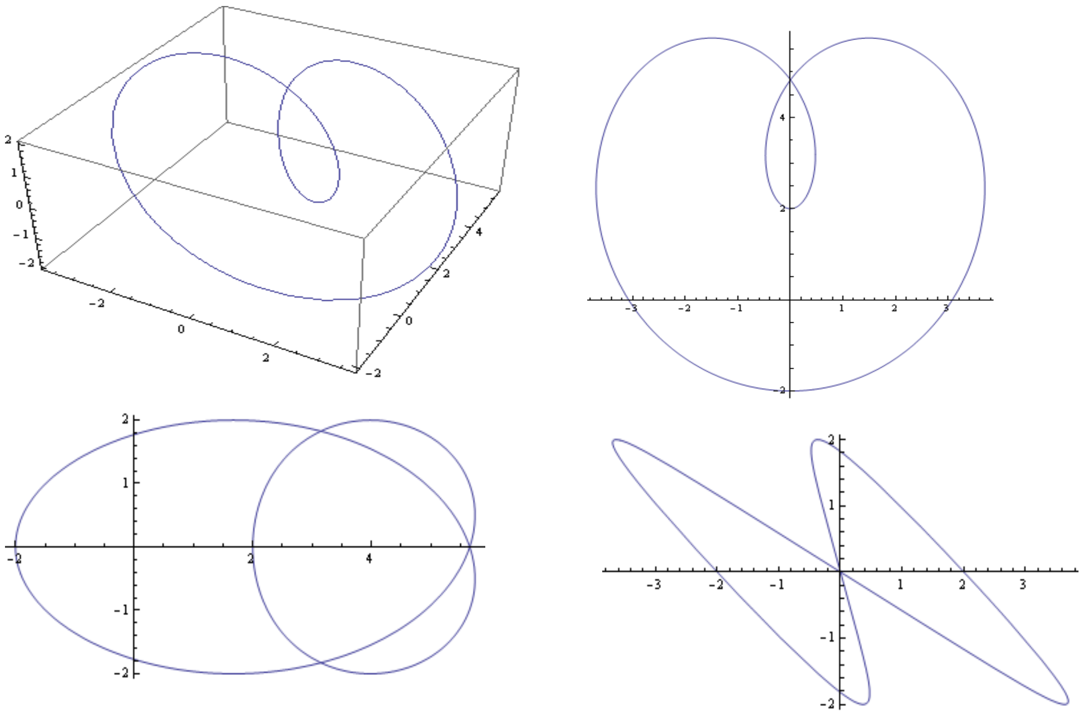

Figure 4. (Top-Bottom, Left-Right) Toroidal Curve $\mathtt{torusgen}[t, t, 4\sqrt{2}, 2]$ and its Projections onto the Planes x − y, y − z, and x − z, respectively.

$$\text{Out[5]} = \left\{\left\{2\,\text{Cos}[v] - \frac{2\,\text{Cos}[u]\,\text{Sin}[v]}{1+\text{Sin}[u]^2}, \frac{2\,\text{Cos}[u]\,\text{Cos}[v]}{1+\text{Sin}[u]^2} + 2\,\text{Sin}[v], \frac{2\,\text{Cos}[u]\,\text{Sin}[u]}{1+\text{Sin}[u]^2}\right\}\right\}$$

In[6]:= ParametricPlot3D[torusg[u, v, 2, 2], {u, −π/2, π/2}, {v, 0, 2π}]

Out[6]= *See Figure 3*

3.1.3. Toroidal Curves and their Projections onto the Coordinate Planes

The parametric equations of the toroidal curve obtained by applying the parameterization given by torusgen[u, v, 4√2, 2] ve t → (t, t) are given by:

In[7]:= torusgen[t, t, 4√2, 2]

$$\text{Out[7]} = \left\{\left\{2\,\text{Cos}[t] - \frac{4\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]}{1+\text{Sin}[t]^2}, 2\,\text{Sin}[t] + \frac{4\sqrt{2}\,\text{Cos}[t]^2}{1+\text{Sin}[t]^2}, \frac{4\sqrt{2}\,\text{Cos}[t]\,\text{Sin}[t]}{1+\text{Sin}[t]^2}\right\}\right\}$$

with graphical representation given by:

In[8]:= ParametricPlot3D[torusgen[t, t, 4√2, 2], {t, 0, 2π}]

Out[8]= *See Figure 4 (top-left)*

The graphical representation of the projections of the toroidal curve given by $\texttt{torusgen}[t, t, 4\sqrt{2}, 2]$ onto the coordinate planes is obtained as:

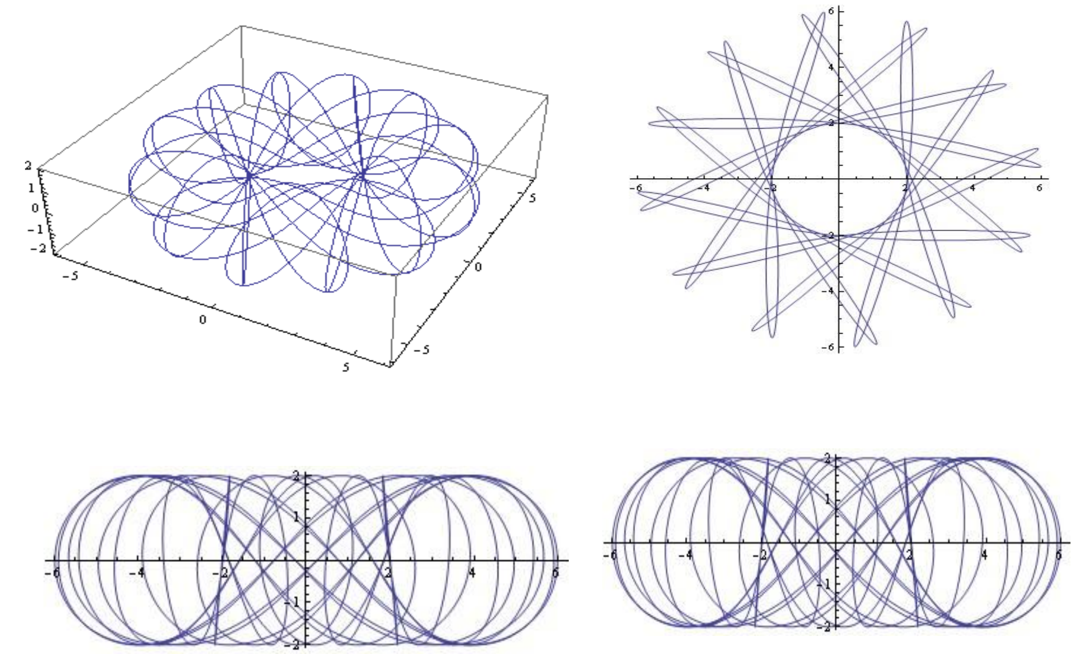

Figure 5. (Top-Bottom, Left-Right) Toroidal Curve $\texttt{torusgen}[12t, t, 4\sqrt{2}, 2]$ and its Projections onto the Planes x − y, y − z, and x − z, respectively.

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In[9]:= ParametricPlot[Delete[
             torusgen[t, t, 4√2, 2], {1, #}, {t, 0, 2π}]&/@{3, 1, 2}
Out[9]= {Fig. 4(top-right), Fig. 4(bottom-left), Fig. 4(bottom-right)}
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Similar steps can be carried out for the toroidal curve given by the same parameterization as applied to the linear curve t → (12t, t), displayed in Figure 5(top-left). The corresponding projections onto the coordinates planes are shown in the other pictures of Figure 5 for the planes x − y, y − z, and x − z, displayed in Figure 5(top-right), (bottom-left), and (bottom-right), respectively.

3.2. Case II: Non-Planar Curves

In this case, we can compute the parameterization of the torus as:

$$\text{In[10]} := \text{torus1}[u\_, v\_] := \left(\begin{pmatrix} \text{Cos}[v] & -\text{Sin}[v] & 0 \\ \text{Sin}[v] & \text{Cos}[v] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2\,\text{Cos}[u] - \frac{4\sqrt{2}\,\text{Cos}[u]\,\text{Sin}[u]}{1+\text{Sin}[u]^2} \\ 2\,\text{Sin}[u] + \frac{4\sqrt{2}\,\text{Cos}[u]^2}{1+\text{Sin}[u]^2} \\ \frac{4\sqrt{2}\,\text{Cos}[u]\,\text{Sin}[u]}{1+\text{Sin}[u]^2} \end{pmatrix} \right)^T$$

$$\text{In[11]} := \text{ParametricPlot3D}\left[\text{torus1}[u, v], \left\{u, \frac{-\pi}{2}, \frac{\pi}{2}\right\}, \{v, 0, 2\pi\}, \text{ViewPoint} \to \{3, 1, -2\}, \text{ViewVertical} \to \{0, 0, -3\}\right]$$

Out [11] = *See Figure 6*

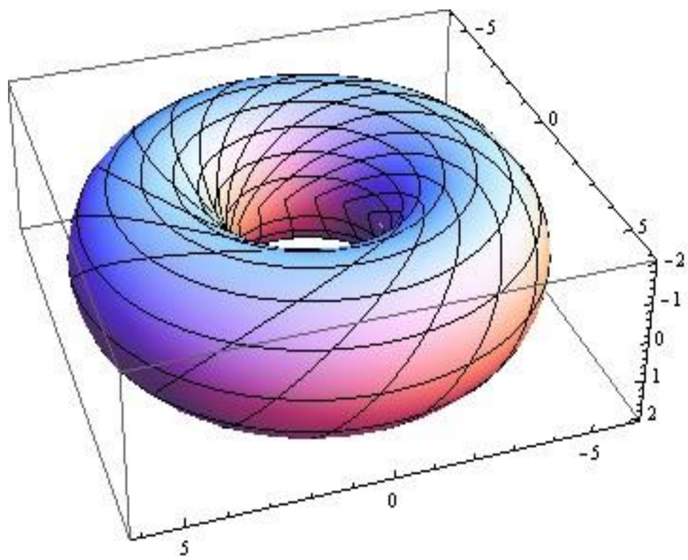


Figure 6. Torus Parameterized through the Non-Planar Curve given by Out [7] for the Domain: −π/2 ≤ u ≤ π/2 ∧ 0 ≤ v ≤ 2π.

$$\text{In[12]} := \text{ParametricPlot3D}\left[\text{torus1}[u, v], \left\{u, \frac{-\pi}{2}, \frac{\pi}{2}\right\}, \left\{v, 0, \frac{7\pi}{4}\right\}, \text{ViewPoint} \to \{3, 1, -2\}, \text{ViewVertical} \to \{0, 0, -3\}, \text{Mesh} \to \{15, 13\}\right]$$

Out [12] = *See Figure 7*

The graphical representation of the toroidal curve for the parameterization torus1[u, v] applied to the linear curve t → (12t, t) is given by:

```
In[13]:= ParametricPlot3D[torus1[12t, t], {t, 0, 2π}]

Out[13]= See Figure 8 (top-left)
```

Similar to the previous case, we can display the projections of the toroidal curve torus1[12t, t]. They are shown in the other three pictures of Figure 8.

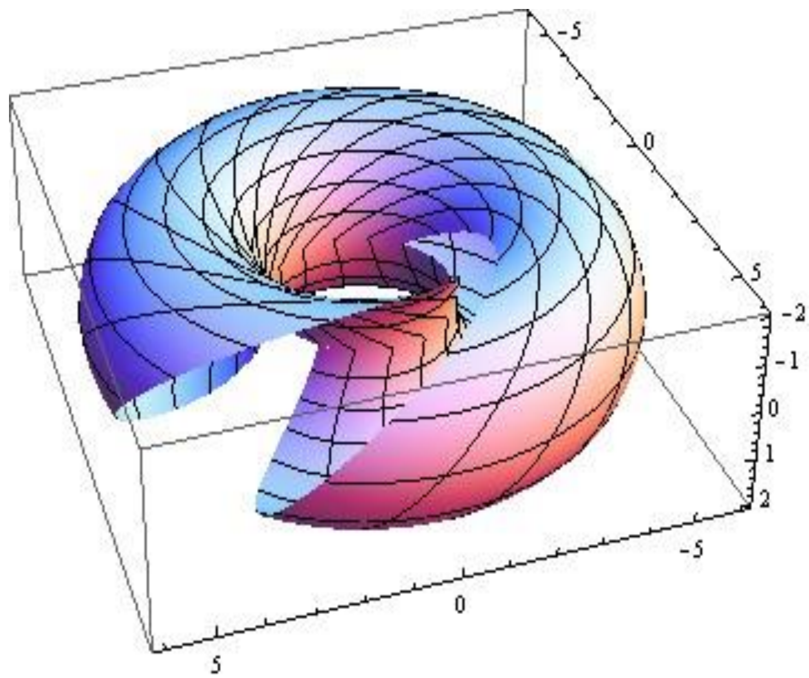
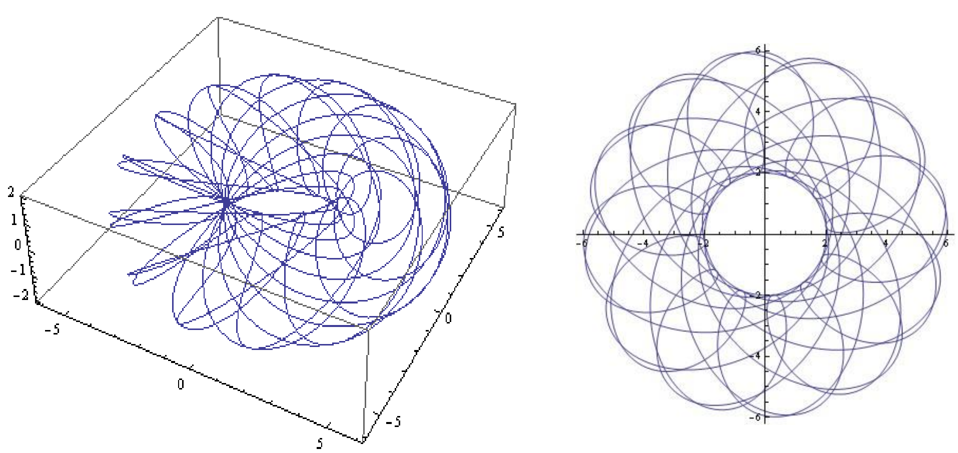


Figure 7. Torus Parameterized through the Non-Planar Curve given by Out [7] for the Domain: −π/2 ≤ u ≤ π/2 ∧ 0 ≤ v ≤ 7π/4



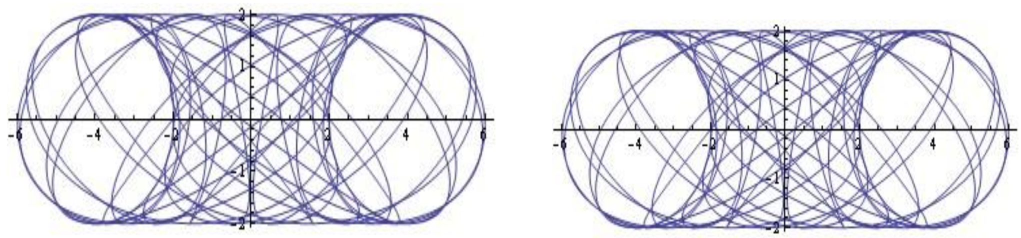


Figure 8. Toroidal Curve torus1 [12t, t] (Top-Left) and its Projections onto the Planes x − y (Top-Right), y − z (Bottom-Left) and x − z (Bottom-Right)

4. Conclusions and Further Remarks

This paper illustrates a symbolic computational approach for the parameterization of surfaces of revolution through curvature-varying curves. The approach relies on the intensive use of the symbolic and numerical features of the popular symbolic computation program Mathematica. This approach is applied to the particular case of a torus, a parametric surface with striking mathematical properties and a wide range of applications in several fields. Our computational analysis shows that the torus can be parameterized not only through circles but also through lemniscates and even non-planar curves. Furthermore, the projections of the curves on the torus derived from such parameterizations onto the coordinates planes exhibit some kind of symmetry with respect to the axis or the origin of coordinates, as shown for instance, in Figures 4, 5, and 8.

All computations in this paper has been done in Mathematica, v7.0 on a 2.9 GHz. Intel Core i7 processor with 8 GB. of RAM. In our experience, this software is an ideal tool for this task, since it provides us with a number of remarkable symbolic, numerical and graphical features, along with a powerful yet simple programming language [3].

Our future work includes the extension of these initial results to other types of surfaces of revolution. The analysis of possible parameterizations of surfaces of revolution through curves exhibiting other interesting geometric properties is also part of our future work.

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