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Dynamic response of a laterally-loaded infinite rigid cylinder embedded in a saturated poroelastic medium

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Abstract

In this paper, an analytical solution for the response of a rigid cylinder embedded in a full-space poroelastic medium subjected to a dynamic lateral load is derived. The problem is idealised as a two-dimensional problem. The solution is obtained using Biot's theory for acoustic waves. In this solution, the displacements of the solid skeleton and the pore pressure are expressed in terms of three scalar potentials. These potential correspond to the wave velocities of the slow and fast compressional wave and to the shear wave. The governing equation for the dynamic motion is expressed in the frequency domain using Fourier transformation and the potentials are shown to be given by Holmholtz equations.

Keywords: Biot's consolidation, dynamic response, poroelasticity

1 Introduction

The dynamic behaviour of rigid inclusions embedded in an elastic saturated porous media provides a useful model for the examination of the dynamic behaviour of structures and foundations embedded in soil media. Models based on poroelasticity theory can be regarded as a first approximation for the investigation of the dynamic soil-structure interaction despite the fact that soil has complicated nonlinear stress-strain characteristics.

The study of the dynamic response of porous medium is often attributed to the two papers by Biot in 1956 (Biot 1956a, 1956b). Since then, several modifications and developments have been introduced. A list of the major contributions in this field can be found in Schanz (2009).

This paper produces a poroelastodynamic analysis of a rigid cylinder deeply embedded in an isotropic elastic medium of infinite extent and subjected to a lateral load. This problem is relevant to a wide range of engineering applications ranging from consolidation around a shaft of a pile foundation to anchor regions that are used to resist uplift loads to in-situ testing of soil using a T-bar full-flow penetrometer. In this paper, a plane-strain section of the cylinder-porous medium system is considered and the problem is analysed in two dimensions. The reference stresses and the pore fluid pressure before loading the cylinder are taken to be zero. The classical sign convention in poroelasticity is adopted where tensile stresses and strains are taken to be positive and a pore pressure, p, corresponds to an isotropic stress level of $-\delta_{ij}p$, where δ_{ij} is the Kronecker delta.

2 Governing equation for dynamic poroelasticity

In polar coordinates (r,θ), the displacements and the pore pressure in the frequency-domain are governed by:

$$\mu\left(\nabla^2\hat{u}_r^s - \frac{\hat{u}_r^s}{r^2} - \frac{2}{r^2}\frac{\partial\hat{u}_\theta^s}{\partial\theta}\right) + (\lambda+\mu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\hat{u}_r^s\right) + \frac{1}{r}\frac{\partial\hat{u}_\theta^s}{\partial\theta}\right) - (\alpha-\beta)\frac{\partial\hat{p}}{\partial r} + \omega^2(\rho_b - \beta\rho_f)\hat{u}_r^s = 0 \tag{1}$$

$$\mu\left(\nabla^2\hat{u}_\theta^s - \frac{\hat{u}_\theta^s}{r^2} + \frac{2}{r^2}\frac{\partial\hat{u}_r^s}{\partial\theta}\right) + (\lambda+\mu)\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\hat{u}_r^s\right) + \frac{1}{r}\frac{\partial\hat{u}_\theta^s}{\partial\theta}\right) - (\alpha-\beta)\frac{1}{r}\frac{\partial\hat{p}}{\partial\theta} + \omega^2(\rho_b - \beta\rho_f)\hat{u}_\theta^s = 0 \tag{2}$$

$$\frac{\beta}{\omega^2\rho_f}\nabla^2\hat{p} + \frac{\phi^2}{R}\hat{p} + (\alpha-\beta)\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\hat{u}_r^s\right) + \frac{1}{r}\frac{\partial\hat{u}_\theta^s}{\partial\theta}\right) = 0 \tag{3}$$

where $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$ is the Laplacian operator in polar coordinates, λ and μ are Lamé's constants for the porous elastic skeleton, p is the pore fluid pressure and u_r^s and u_θ^s are the radial and the circumferential displacements of the solid skeleton; respectively, ρ_f is the fluid density and ρ_b is the bulk density given by $\rho_b = (1-\phi)\rho_s + \phi\rho_f$ where ρ_s is the density of the solid, ϕ is the porosity, R is a parameter calculated from the bulk moduli of the constituents and characterises the coupling between the fluid and the solid, $w_i = \phi(u_i^f - u_i^s)$ is the seepage displacement, where u_i^f and u_i^s are the fluid and the solid displacement respectively, ω is the frequency and ^ donates Fourier transformation with respect to time defined as:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \tag{4}$$

and the parameter β is given by:

$$\beta = \frac{\omega^2\phi^2\kappa\rho_f}{\omega^2\kappa\left(\rho_a + \phi\rho_f\right) - i\omega\phi^2} \tag{5}$$

where κ denotes the permeability and ρ_a is the apparent density describing the dynamic interaction between the fluid and the skeleton.

The boundary value problem can be formulated by treating the displacement components and pore pressure as dependent variables. However, the use of special displacement and stress functions representations can result in a reduction in the number of dependent variables as demonstrated by Norris (1985), Zimmerman and Stern (1993) and Lu and Jeng (2007). Using Helmholtz decomposition technique, the radial and the circumference displacements can be derived from three displacement functions, χ_1, χ_2 and ψ corresponding to fast and slow compression waves and shear wave, respectively:

$$\hat{u}_r^s = \frac{\partial}{\partial r}\left(\hat{\chi}_1 + \hat{\chi}_2\right) + \frac{1}{r}\frac{\partial\hat{\psi}}{\partial\theta} \tag{6}$$

$$\hat{u}_\theta^s = \frac{1}{r}\frac{\partial}{\partial \theta}\left(\hat{\chi}_1 + \hat{\chi}_2\right) - \frac{\partial \hat{\psi}}{\partial r} \tag{7}$$

Following Lu and Jeng (2007), the pore fluid pressure can be related to displacement functions by:

$$\hat{p} = A_1 \nabla^2 \hat{\chi}_1 + A_2 \nabla^2 \hat{\chi}_2 \tag{8}$$

where A_1 and A_2 are frequency-dependent constants

By substituting equations 6 , 7 and 8 into equations 1,2 and 3 it could be shown that

$$\begin{aligned} &\nabla^2 \hat{\chi}_1 - q_1^2 \hat{\chi}_1 = 0 \\ &\nabla^2 \hat{\chi}_1 - q_2^2 \hat{\chi}_1 = 0 \\ &\nabla^2 \hat{\psi} - q_3^2 \hat{\psi} = 0 \end{aligned} \tag{9}$$

where

$$\begin{aligned} q_1 &= \sqrt{\frac{-\omega^2 (\rho_b - \beta \rho_f)}{(\lambda + 2\mu) - A_1(\alpha - \beta)}} \\ q_2 &= \sqrt{\frac{-\omega^2 (\rho_b - \beta \rho_f)}{(\lambda + 2\mu) - A_2(\alpha - \beta)}} \\ q_3 &= \sqrt{\frac{-\omega^2 (\rho_b - \beta \rho_f)}{\mu}} \end{aligned} \tag{10}$$

and

$$A_{1,2} = \frac{1}{2(\alpha - \beta)\rho_f \phi^2} \begin{pmatrix} (\lambda + 2\mu)\rho_f \phi^2 - R\left(\alpha^2 \rho_f + \beta(\rho_b - 2\alpha\rho_f)\right) \\ \pm \sqrt{\left((\lambda + 2\mu)\rho_f \phi^2 - R\left(\alpha^2 \rho_f + \beta(\rho_b - 2\alpha\rho_f)\right)\right)^2 + 4R(\alpha - \beta)^2 (\lambda + 2\mu)\rho_f^2 \phi^2} \end{pmatrix} \tag{11}$$

3 Solution of the governing equation for dynamic poroelasticity

Pore fluid pressure, stresses and displacements should be finite and reduce to zero as $r \to \infty$. If a cylinder of radius r_0 is taken to be fully-bonded to the surrounding elastic medium, then the relation between the radial and the circumferential components of the displacements at the interface between the rigid cylinder and the elastic medium is governed by:

$$\hat{u}_r \sin\theta + \hat{u}_\theta \cos\theta = 0 \qquad at \quad r = r_0 \tag{12}$$

The component of tractions acting on any circumference which encloses the rigid cylinder and centred about its origin in the direction of the applied force F (Fig. 1) is governed by:

$$\int_0^{2\pi} r_0 \left(\hat{\sigma}_r \cos\theta - \hat{\sigma}_{r\theta} \sin\theta\right) \mathrm{d}\theta + \pi r_0^2 \rho_r \omega^2 \hat{\xi} = -\hat{F} \tag{13}$$

where σ is the stress, ρ_r is the density of the rigid cylinder

For impermeable cylinder:

$$\hat{w}_r = \frac{\beta}{\omega^2 \rho_f} \left(\frac{\partial \hat{p}}{\partial r} - \omega^2 \rho_f \hat{u}_r^s \right) = 0 \qquad at \quad r = r_0 \tag{14}$$

The translation of a circular rigid disk can be expressed in polar coordinates as:

$$\begin{aligned} \hat{u}_r(r,\theta,\omega) &= \hat{u}_r^*(r,\omega)\cos\theta \\ \hat{u}_\theta(r,\theta,\omega) &= \hat{u}_\theta^*(r,\omega)\sin\theta \end{aligned} \tag{15}$$

From equation 9, it can be shown that:

$$\begin{aligned} \hat{\chi}_1 &= BK_1(q_1 r)\cos\theta \\ \hat{\chi}_2 &= CK_1(q_2 r)\cos\theta \\ \hat{\psi} &= DK_1(q_3 r)\sin\theta \end{aligned} \tag{16}$$

$K_v(x)$ is modified Bessel function of second kind of order v.

The coefficients B C and D are determined from the boundary conditions (equations 12,13 and 14).

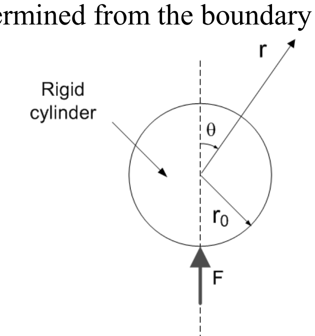

Figure 1, Nomenclature for a rigid cylinder under lateral load

4 Results and discussion

Any analytical solution for a laterally loaded cylinder must be applied over a wide range of geometries and material properties. It is helpful to form dimensionless groups of relevant parameters, rather than investigate how the solution is affected by the variation of each individual material parameter. Following the technique of dimensional analysis, the stress, displacement and the pore fluid pressure at any fixed location can be expressed as:

$$\left\{\frac{|\mathbf{u}|\mu}{|F|},\frac{|p|r_0}{|F|},\frac{|\boldsymbol{\sigma}|r_0}{|F|}\right\}=\mathrm{f}\left(\omega \mathrm{r}_0\sqrt{\frac{\rho_b}{\mu}},\frac{\lambda}{\mu},\frac{R}{\mu},\frac{\rho_r}{\rho_f},\frac{\rho_b}{\rho_f},\frac{\rho_a}{\rho_f},\phi,\alpha,\kappa\frac{\sqrt{\mu\rho_b}}{\mathrm{r}_0}\right) \tag{23}$$

Figure 2 shows the response of a cylinder subjected to a harmonic load of the form:

$$F=F_n e^{i\omega t} \tag{24}$$

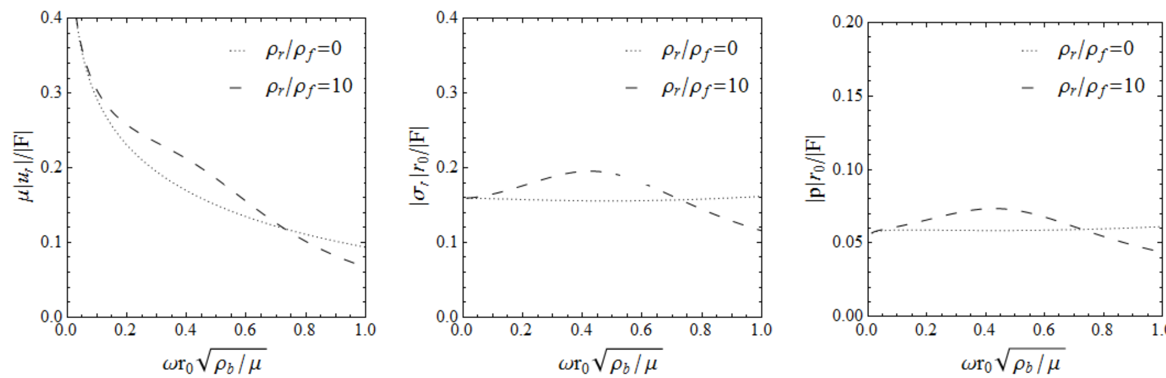

Figure 2, Effect of the density of the rigid cylinder

The analysis was carried out using Wolfram MATHEMATICA 7.0 software. The analysis was carried out with the parameters shown in Table 1. The results are shown for two cylinder/fluid density of 0 and 10. These values were selected to represent extreme cases. The results are plotted for a point adjacent to the cylinder (r = r0) located in the direction of the load (θ = 0°). These results show that in the case of ρ_r/ρ_f=10, the total stress and the pore fluid pressure had a peak value at a normalised frequency of about 0.42.

Table 1 Parameters used in the analysis

$\frac{\lambda}{\mu}$	$\frac{R}{\mu}$	$\frac{\rho_b}{\rho_f}$	$\frac{\rho_a}{\rho_f}$	ϕ	α	$\kappa\frac{\sqrt{\mu\rho_b}}{\mathrm{r}_0}$
0.667	0.1503	2.05	0.0	0.3	0.95	10^{-4}

Figure 3 shows the response for two material properties (λ/μ = 4 and λ/μ = 2/3, corresponding to Poisson's ratio ν = 0.4 and 0.2). The rigid cylinder is taken to be weightless ρ_r/ρ_f=0. This figure shows that the magnitude of the total radial stress is not affected by λ/μ ratio, while the pore fluid pressure and the radial displacement increases with the decrease of λ/μ. Figure 4 shows that the permeability has a significant effect on the pore fluid pressure in low frequencies. However, it is effect is less significant in high frequencies. The displacement and the total radial stress are not affected by the permeability of the porous medium.

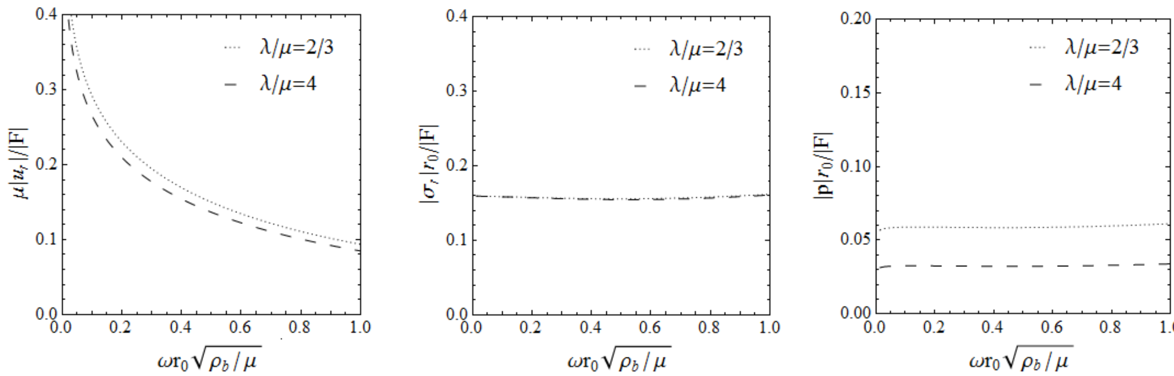

Figure 3, Effect of Poisson’s ratio

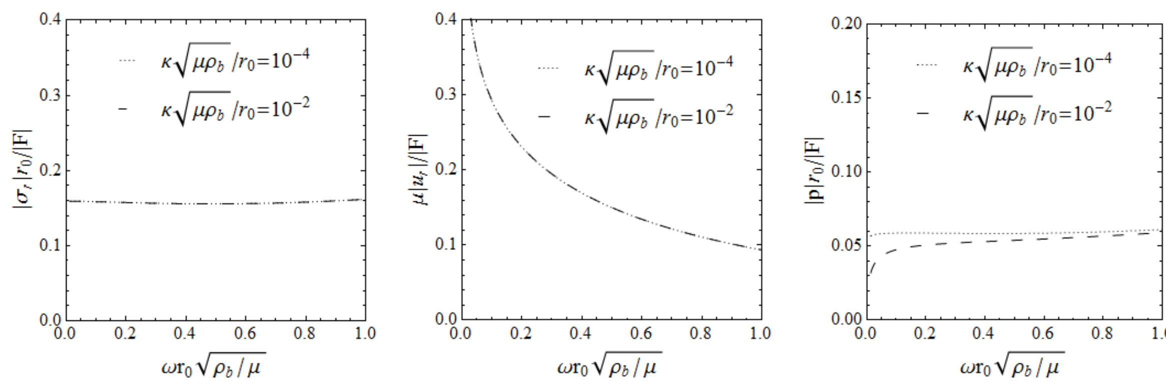

Figure 4, Effect of permeability

5 Conclusions

An analytical solution for the response of a laterally loaded rigid cylinder has been derived using poroelastodynamic formulation. The solution is based on a two-dimensional plane-strain idealisation. Exact analytical solutions for pore fluid pressure, stresses and displacements in the frequency domain are obtained using the stress function technique.

References

BIOT, M. A., 1956, Theory of Propagation of Elastic Waves in a Fluid-Saturated Porous Solid. I. Low-Frequency Range, *J. Acoust. Soc. Am.*, **28**, pp. 168–178.

BIOT, M. A., 1956, Theory of Propagation of Elastic Waves in a Fluid-Saturated Porous Solid. II. Higher Frequency Range, *J. Acoust. Soc. Am.,* **28**, pp. 179–191.

Lu J-F and JENG D-S (2007): A half-space saturated poroelastic medium subject to a moving point load. *International Journal of Solid and Structures*, 44(2), 573–586.

NORRIS, A. N., 1985, Radiation From a Point Source and Scattering Theory in a Fluid-Saturated Porous Solid, *J. Acoust. Soc. Am.,* **77**, pp. 2012–2023.

SCHANNZ M, 2009.Poroelastodynamics: Linear Models, Analytical Solutions, and Numerical Methods, *ASME Applied Mech. Review* **62** 3, S. 030803-1 - 030803-15

ZIMMERMAN, C., STERN, M., 1993. Scattering of plane compressional waves by spherical inclusions in a poroelastic medium. *J. Acoust. Soc. Am.* 94, 527–536.