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An efficient formulation for calculating the vibration response of piled foundations due to excitation by an underground railway

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Abstract

The propagation of vibration from underground railways is an important environmental issue, as it results in structure-borne noise and vibration which is both irritating to occupants and disruptive to operations. Due to the complexity of the railway-soil-foundation-building system, very few comprehensive models exist for the prediction of surface noise and vibration levels. Existing models have exceedingly-long computation times, making them unsuitable for evaluating design changes. For this reason, a model is sought which captures the essential dynamic behaviour of the system whilst using a minimal number of system parameters and requiring an under-five-minute runtime.

The model presented in this paper is for a piled foundation, embedded in soil and subject to an incident wavefield generated by the existing Pipe-In-Pipe underground railway model. The piles are modelled as columns in axial vibration, and it is shown that this formulation can be extended to include Euler beams in lateral vibration. Whilst this model may neglect small amounts of cross-coupling between different vibration directions, the overall behaviour of the piles can be captured sufficiently using these one-dimensional models for axial and lateral vibration. The soil is modelled as a linear, elastic continuum and has the geometry of a thick-walled cylinder with an infinite outer radius and an inner radius equal to the radius of the pile. Pile-soil-pile interactions are accounted for by using the superposition of interaction factors; and the method of joining subsystems is used to incorporate the incident wavefield generated by the underground railway into the pile model.

The resulting model is validated by comparison with those derived using complex numerical methods, namely FEM-BEM and BEM at all possible stages. Excellent agreement is observed in both the axial response of a single pile to a pile-head loading, and in the calculated interaction factors.

Keywords: pile foundation, underground railway, ground-borne vibration, pile-soil-pile interaction

1 Introduction

The vibration response of piled foundations due to ground-borne vibration produced by an underground railway is a largely-neglected area in the field of structural dynamics. However, this continues to be an important aspect of research as it is currently believed that the presence of piled foundations will have a significant influence on the propagation and transmission of the wavefield produced by the underground railway. It is thought that the lack of such a model is due to the complexity of modelling such a system, particularly the large number of elements required (and subsequent increase in computation time) when using numerical finite element or boundary element models.

This paper presents the formulation of an efficient, 3D model for calculating the response of a piled foundation subject to excitation from an underground railway. The model is presented in three sections: the formulation of a single pile model; calculation of pile-soil-pile interactions; and the response of a pile to an incident wavefield.

2 Single pile model

The formulation of the single pile model begins with consideration of an infinitely-long pile embedded in a homogeneous, elastic fullspace. The pile is modelled as a column in axial vibration or an Euler beam in lateral vibration. The soil is modelled as an elastic continuum with outer radius of infinity and inner radius equal to the pile radius. The derivation of the equations for such a continuum are given in Forrest (1999), and the matrices $[\boldsymbol{T}_\infty]$ and $[\boldsymbol{U}_\infty]$ referred to here are the same as the matrices of the same name presented in Forrest. For the sake of brevity only the equations for a pile undergoing axial vibration are presented here. The equations for a pile undergoing lateral vibration are obtained by replacing the column with an Euler beam, and then following a similar formulation method to the axial case. The infinitely-long pile is adapted to model a finite pile in Section 2.2 using the mirror-image method.

2.1 Axial vibration of an infinite pile

Consider an infinite column of radius a undergoing axial vibration, defined by polar coordinates (r, θ, z) and embedded in an infinite elastic continuum. The displacement of the column at any given distance z along the column is invariant over the cross-sectional area, thus the displacement of the column at any point can be defined by the displacement in the z-direction: $\widetilde{U}_z$. The displacements in the tangential and radial directions, $\widetilde{U}_\theta$ and $\widetilde{U}_r$ respectively, are zero along the column (Poisson effects are neglected). The column is assumed to be perfectly welded to the continuum at the column-continuum interface, so the displacement on the inner surface of the elastic continuum is defined only by $\widetilde{U}_z$, as $\widetilde{U}_\theta$ and $\widetilde{U}_r$ are zero. The elastic continuum equations presented in Forrest (1999) can be used to obtain the stresses $\{\tilde{T}_{rz}, \tilde{T}_{r\theta}, T_{rr}\}^T$ acting at the inner surface of the soil cylinder due to some arbitrarily imposed displacement distribution $\widetilde{U}_z$:

$$\begin{Bmatrix} -\tilde{T}_{rz} \\ -\tilde{T}_{r\theta} \\ \tilde{T}_{rr} \end{Bmatrix} = [\boldsymbol{T}_\infty]_{r=a} [\boldsymbol{U}_\infty]_{r=a}^{-1} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \widetilde{U}_z \tag{1}$$

As the majority of motion is occurring in the z -direction, the magnitude of the stresses in the r and θ directions is negligible and therefore these stresses can be ignored. The force $\tilde{F}_{rz}$ arising from the stress $-\tilde{T}_{rz}$ acting at the column-continuum interface is obtained by integrating over the column circumference for a unit length in the z-direction:

$$\tilde{F}_{rz} = \int_0^1 \int_0^{2\pi} -\tilde{T}_{rz} a d\theta dz = -2\pi a \tilde{T}_{rz} \tag{2}$$

Thus the force acting at the column-continuum interface $\tilde{F}_{rz}$ can be related to the displacement distribution imposed at the column-continuum interface $\widetilde{U}_z$ by a function that is denoted here as the vertical stiffness of the elastic continuum, $K(\xi)$:

$$\tilde{F}_{rz} = 2\pi a [1 \quad 0 \quad 0] [\boldsymbol{T}_\infty]_{r=a} [\boldsymbol{U}_\infty]_{r=a}^{-1} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \widetilde{U}_z = K(\xi) \widetilde{U}_z \tag{3}$$

Where ξ is the wavenumber; the Fourier pair of the variable z.

As the elastic continuum (the soil) is now represented by a single stiffness function, the soil-pile model can be represented as an infinite column subject to a unit harmonic excitation at $z = 0$ and some force distribution $f_{rz}(z)$ imposed by the soil on the pile, where $f_{rz}(z)$ is the Fourier transform pair of $\tilde{F}_{rz}$. Using the standard equation for the axial vibration of an infinite column with Young's modulus E, cross-sectional area A, and mass per unit length m', the equation for this system is:

$$-EA\frac{\partial^2 u}{\partial z^2} + m'\frac{\delta^2 u}{\delta t^2} = \delta(z) - f_{rz}(z) \tag{4}$$

Application of the Fourier transform, substitution of Eq.3 and rearrangement of the resulting equation gives the displacement of the pile in the wavenumber-domain:

$$\tilde{U}(\xi) = \frac{1}{EA\xi^2 + K(\xi) - m'\omega^2} \tag{5}$$

The displacements $\{\tilde{U}_z, \tilde{U}_\theta, \tilde{U}_r\}^T_{r=R}$ at some radius R elsewhere in the soil can be calculated using the equation:

$$\begin{Bmatrix} \tilde{U}_z \\ \tilde{U}_\theta \\ -\tilde{U}_r \end{Bmatrix}_{r=R} = [\boldsymbol{U}_\infty]_{r=R}[\boldsymbol{U}_\infty]^{-1}_{r=a}\begin{Bmatrix} \tilde{U} \\ 0 \\ 0 \end{Bmatrix}_{r=a} \tag{6}$$

The displacements in space $\{u(z,t), v(z,t), w(z,t)\}^T$ are obtained by applying an inverse Fourier transform to the wavenumber-domain displacements:

$$\begin{Bmatrix} u(z,t) \\ v(z,t) \\ w(z,t) \end{Bmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{Bmatrix} \tilde{U}_z \\ 0 \\ -\tilde{U}_r \end{Bmatrix} e^{i\xi z} d\xi e^{i\omega t} \tag{7}$$

2.2 *Making finite piles*

In order to model a single, finite pile of length L, it is necessary to introduce a free surface at $z = 0$, and a free end at $z = L$ into the infinite pile-soil model. The mirror-image method, in which scaled forces and moments are applied at two points equidistant from the required end-condition position can be used to simulate zero-force and zero-moment conditions.

Before presenting the procedure for applying the mirror-image method, a caveat regarding must be made. Whilst it can be shown that exact solutions are obtained using the mirror-image method on simple one-dimensional systems such as columns or beams, when this method is applied to systems with three-dimensional stress states, errors do arise. This is because the mirror-image method does not produce a stress field which completely satisfies the traction-free boundary conditions at the free surface. When mirror-image vertical forces are applied, the vertical stresses at the free surface are zero, but there exists some residual shear stress at the surface. Similarly, when mirror-image lateral forces are applied the shear stresses at the free surface are zero, but there exists some residual vertical stress at the surface (Sen et al., 1985). Given that the mirror-image method is being applied here to systems which are being excited primarily in one direction, it is estimated that the magnitude of the error introduced by using this method is small. However, these errors may become significant when displacements perpendicular to the primary direction of excitation are calculated on the free surface.

The procedure for applying the mirror-image method to the infinite pile-soil system is:

1. Apply twice the unit force to the infinite column at $z = 0$ and calculate the response of the pile system using Eq.5.
2. Use Eq. 7 to transform this response into the z-domain, then calculate the force $EA\frac{du}{dz}$ at $z = L$.
3. Apply scaled mirror image forces $P^* = -EA\left(\frac{du}{dz}\right)_{z=L}$ to the infinite column at $z = L$ and $z = -L$, and calculate the response of the column-soil system in the z-domain using Eqs.5&7.

4. Superimpose the displacement response calculated in step 2 and the displacement response from the mirror-image forces.

2.3 Results of the single pile model

The finite, single pile model that has been derived here will be compared with the output of more complex numerical methods, namely those employing FEM and BEM. Two such models are presented for comparison: a coupled BEM-FEM model developed by Stijn (2009) and a BEM model developed by Talbot (2001). Figure 1 shows the magnitude and phase of the driving point response of a 20m pile to unit harmonic excitation in the z-direction as a function of dimensionless frequency $a_0 = \frac{\omega a}{c_s}$, where c_s is the shear wave speed in the soil. It can be seen from this figure that the response of the pile is relatively invariant with frequency, and that there is good agreement between all the models. The magnitude deviation between the BEM and the single pile model formulated in this paper has been shown to be eliminated as more elements are used to model the pile boundary (Coulier, 2009).

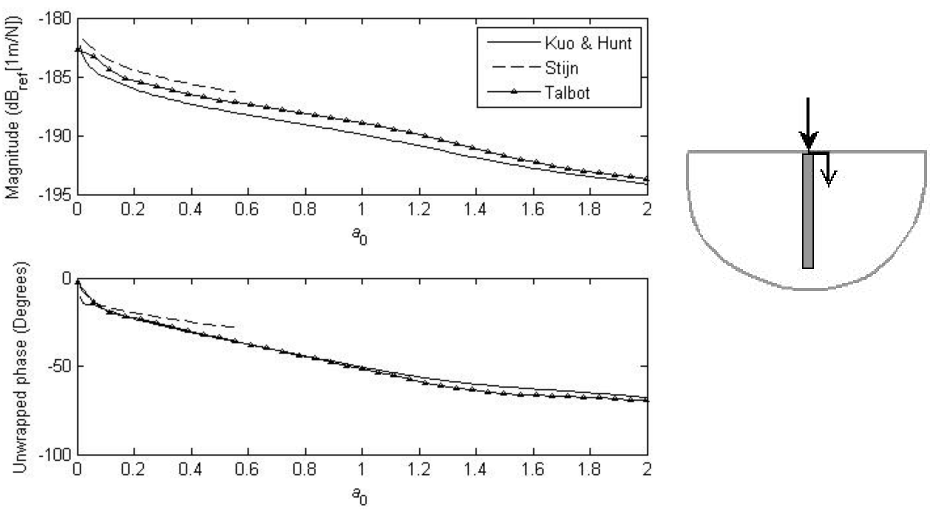


Figure 1. The vertical driving point response of a pile subjected to unit harmonic excitation in the z-direction.

3 Interaction between neighbouring piles

When modelling pile groups, it is necessary to account for the interactions occurring between neighbouring piles, also known as pile-soil-pile interaction (PSPI). These interactions are commonly quantified using the interaction factor: a measure of how the excitation of one pile affects the pile-head displacement of a neighbouring pile. The interaction factor is an important variable to calculate, as it has been shown that the response of a pile in a pile group can be calculated by superimposing the effect of each neighbouring pile (Kaynia, 1982). The method used here is based on the theory of joining subsystems: the reader is referred to a previous paper by Kuo & Hunt (2009) for a detailed explanation of this method.

The two subsystems are illustrated in Figure 2. Subsystem A is represented by a finite-length pile undergoing harmonic excitation $X_1(\omega)$ at the pile head and a nearby cylindrical void, and subsystem B is represented by a column (or a Euler beam in the case of lateral vibration) to be coupled to the void surface of subsystem A. The displacement output $Y_2(\omega)$ is to be calculated at the head of the pile represented by subsystem B, therefore the governing equation for this system is:

$$Y_2(\omega) = [I + A_{33}B_{33}^{-1}]^{-1}A_{31}X_1(\omega) \tag{8}$$

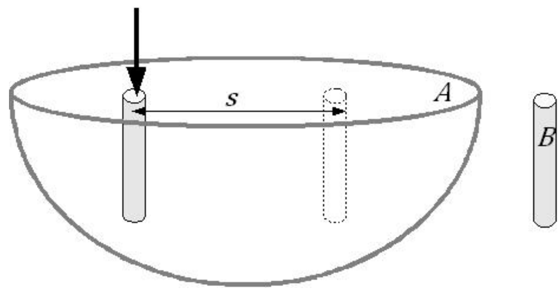


Figure 2. The two separate subsystems which are joined together to form a two-pile system.

The frequency response function $A_{31}(\omega)$ relates the response at the cylindrical void to the excitation on the pile head $X_1(\omega)$. This can be approximated by using the finite, single pile model to calculate the response at distance s from the original pile.

The frequency response function $A_{33}(\omega)$ relates the response at the cylindrical void to forces acting on the cylindrical void. This response has already been calculated for axial vibration in the form of the vertical stiffness of the elastic continuum $K(\xi)$. Assuming the original pile has little influence on the void, the frequency response function $A_{33}(\omega)$ is the inverse of this stiffness.

The frequency response function $B_{33}(\omega)$ relates the response of the column to forces acting on the column. No approximation is required, as the exact solution is obtained using the equations for the vibration of a column. The equation for axial vibration of a semi-infinite column is:

$$B_{33}(\omega) = \frac{2}{EA\xi^2 - m'\omega^2} \tag{9}$$

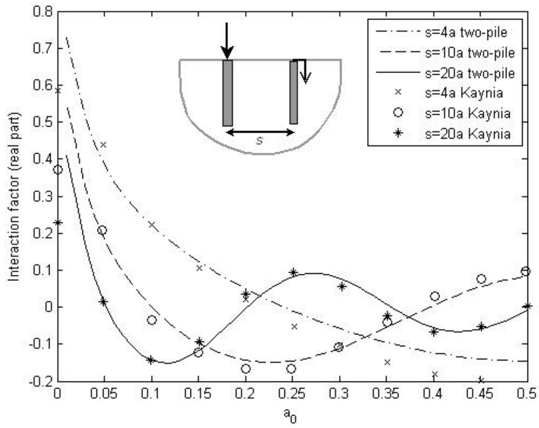
By substitution of these frequency response functions into Eq.8 the response of a semi-infinite pile to excitation by a finite-length pile can be determined. In order to obtain the interaction factor, the mirror-image method must be applied to the semi-infinite pile. The method used to calculate this finite-length, two-pile model is summarised as follows:

1. Create a finite pile and calculate the displacement at a distance s from the pile axis using the method detailed in Section 2.
2. Attach pile 2 using the method of joining subsystems. The displacements propagated from pile 1 represent those produced in a halfspace, therefore pile 2 is a semi-infinite column.
3. Convert semi-infinite pile 2 to a finite pile by applying scaled mirror-image forces $P^* = -EA\left(\frac{du}{dz}\right)_{z=L}$ at $z = L$ & $z = -L$ such that the resulting force $EA\frac{du}{dz}$ at $z = L$ is zero, as in Section 2.2.

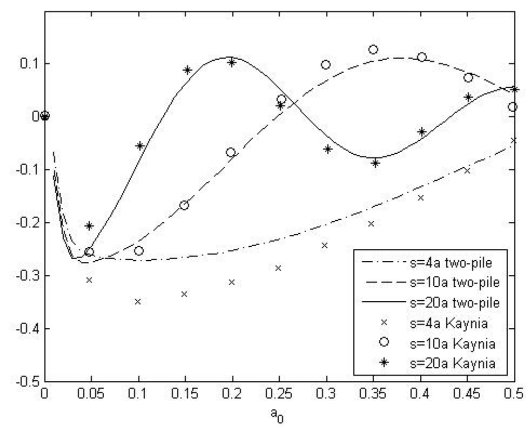
The interaction factor is then obtained by dividing the displacement at the head of pile 2 by the static head displacement of a single pile. It is expected that the application of the additional mirror-image forces will result in some additional displacement being radiated to pile 1, and thus may introduce some inaccuracies into the model. However, the magnitude of these inaccuracies is expected to be small, and to decrease with increasing pile separation distance.

3.1 Results of the two-pile model

The interaction factors calculated here are compared with those obtained by Kaynia (1982) using a dynamic stiffness matrix method. The result in Figure 3 shows the real and imaginary parts of the axial vibration interaction factors calculated at $s = 4a$, $s = 10a$ and $s = 20a$, and plotted as a function of dimensionless frequency. The interaction factors calculated using the two-pile formulation show good agreement with those obtained using the DSM formulation, particularly at large separation distances where the error from the use of the mirror-image method is small.



(a)



(b)

Figure 3. (a) Real part; and (b) imaginary part of the axial response of a finite pile using the two pile model.

4 Piles subjected to vibration from underground railways

The model of a single pile subjected to vibration from underground railways is a similar problem to the calculation of pile-soil-pile interaction factors in all aspects except for the source of the incoming vibration. Previously, the incident wavefield was generated by the excitation of a neighbouring pile, but now the incident wavefield is generated by the passage of a train through a tunnel. The model used here to calculate the incident wavefield is the Pipe-in-Pipe (PiP) model, a 2.5D model of an infinite, thin-shelled cylinder (representing the tunnel) embedded in a homogeneous halfspace. The passage of a train is simulated using a pull-through, unit-roughness spectrum. Further information on this model is available from www.pipmodel.com.

For the pile subjected to axial vibration from underground railways, the incident wavefield is calculated in the form of vertical soil displacements at a series of points along the length of the pile. This represents the input $A_{31}(\omega)X_1(\omega)$ in Eq.8. The spacing and number of these points are chosen to correspond with the discretisation of the semi-infinite pile model in the space domain. This discretisation of 1024 points, each 0.25m apart, has been selected to ensure at least five calculation points per wavelength. The efficiency of this model is limited by the significant computation time involved in using the PiP model to obtain displacements at each of the points along the pile. As the finite pile stretches over only a small portion of the range covered by the points, it is conceivable that not all points are required to obtain a representation of the finite pile. It is observed that the strong damping influence of the soil causes many of the points at a considerable distance from the finite pile section to have no influence on the response of the finite pile. Comparison of the displacements calculated using all 1024 PiP output points and the displacements calculated using only a smaller portion of the PiP output points showed that accuracy is preserved by including only points stretching over a length of three-times the finite pile length L.

4.1 Results of the pile subjected to vibration from underground railways

Using the formulation detailed in Section 3 with the input $A_{31}(\omega)X_1(\omega)$ replaced by the PiP vertical soil displacements, the pile-head displacement of a single pile subjected to vibration from underground railways can be calculated. The results are presented in terms of insertion gain, which measures the increase in vibration levels when a pile is incorporated into the underground railway environment. In the example shown in Figure 4, the insertion gain is calculated for a 20m pile located 10m from the tunnel in the vertical direction. The attenuation achieved by the incorporation of this pile is highly dependent on frequency, and has a mean value of 10.4dB. The mean magnitude of this insertion gain for a single pile implies that a high degree of inaccuracy exists in any surface vibration prediction model that does not include piled foundations.

It should be noted that although the results presented here are those for a single pile subject to an incident wavefield generated by an underground railway, this method can be applied to wavefields generated by other sources, and can also be easily extended to consider the response of a pile group. By calculating the response of each individual pile to the incident wavefield and then accounting for the interactions between neighbouring piles using the superposition of interaction factors, the response of a pile group is determined. This formulation is highly efficient, incorporating the effects of additional piles with very little increase in computation time and thereby overcoming the computational power limitations encountered when modelling pile groups using numerical methods.

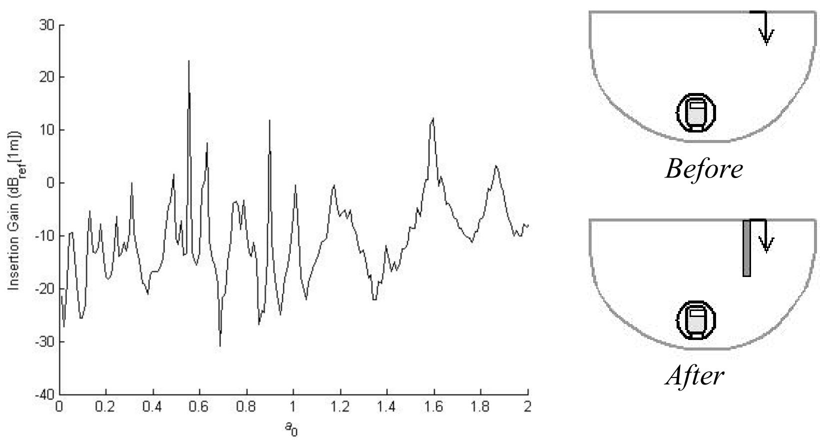


Figure 4. Vertical insertion gain for a 20m pile located 10m from an underground railway.

5 Conclusions

This paper has outlined the formulation of an efficient model for the vibration of piled foundations due to inertial and underground-railway-induced loadings. Results have been computed for a single pile subject to an inertial loading, pile-soil-pile interactions, and a single pile subjected to excitation from an underground railway. Comparisons with models derived using complex numerical methods such as BEM or FEM-BEM have shown excellent agreement. This model represents a significant improvement on previous pile models, not only because it presents a previously-unavailable solution for calculating the response of pile groups to an incident wavefield generated by an underground railway, but also because this efficient model overcomes the computational power limitations encountered when modelling pile groups using numerical methods.

References

FORREST, J.A., 1999. *Modelling of ground vibration from underground railways*. PhD thesis, Cambridge University, UK.

SEN, R., DAVIES, T.G. and BANERJEE, P.K., 1985. Dynamic analysis of piles and pile groups embedded in homogeneous soils. *Earthquake Engineering & Structural Dynamics*, 13(1), 53-65.

TALBOT, J.P., 2001. *On the performance of base-isolated buildings: a generic model*. PhD thesis, Cambridge University, UK.

STIJN, F., 2009. *Reference calculation for pile-soil interaction*. Personal Communication.

COULIER, P., 2009. *BEM investigations into the vibration of pile groups*. Personal Communication.

KUO, K.A. and HUNT, H.E.M., 2009. Back to the future: how dynamic pile models published 30 years ago can be used to answer questions on underground railway design. *In: Noise and Vibration: Emerging Methods*, Oxford, UK.

KAYNIA, A.M., 1982. *Dynamic stiffness and seismic response of pile groups*. PhD thesis, Massachusetts Institute of Technology, USA.