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What is a solution to a differential equation?

Introduction

The answer to any meaningful question depends on the question. This note is an attempt to explain how this (obvious?) principle manifests itself in the study of differential equations.

For the moment, we can regard a differential equation as an equation involving an unknown X, and we say that we have solved the equation if we have found some quantity which “fits” the conditions of the equation. As a non-technical example, let our equation be

$X =$  . (1)

To solve this, we must find an X such that “X is an apple”. Unfortunately, this still leaves some room for creative thinking. One answer could be

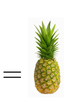
$X =$  .

In some sense this is natural, since the equation (1) then could be interpreted to read “an apple is an apple”, which at least seems to be true. However, a very stringent person may point out that when our solution candidate is inserted into the equation, we get

 $=$  ,

which looks a bit odd. These apples are not that similar! Continuing this line of reasoning will eventually lead us to conclude that the only solution candidate is the *same* apple as the one on the right hand side of (1).

Similarly, if we accept the yellow apple as a solution, then we should accept a red apple as well. Indeed, any apple should be a solution. But where to draw the line in this case, do we accept this

$X =$  ?

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What is a solution

Again, the answer depends on how we interpret (1). If we say that the equation means “X is anything with “apple” in its name”, we are forced to accept the pineapple. At least if we stick to the English language. If we had used French, we would not accept the pineapple, but a potato would be suitable. Thus, after some reflection, this interpretation of (1) does not seem to be very fruitful.

A more reasonable interpretation of (1) is “X is a fruit”. Then both the yellow apple and the pineapple would be reasonable solutions, but a potato would not.

Sometimes, we wish our solution to have some extra property. In many cases such properties come in handy in when we use the solution in other situations. If I look for a solution of (1) which can also be used as a phone, the immediate candidate is

$X =$  .

So we see that depending on how we interpret (1), this equation can have many different solutions. Some, such as “any fruit” are easy to find, others, such as the iphone, are harder to get.

Sometimes it can also be hard to determine whether a given candidate is a solution or not. Is this

a solution? It looks a bit like an apple, but not completely natural. I would estimate that *the probability* of this being an apple is 20%. This leads to another, and more general, concept of a solution. A solution to (1) is a machine, or an algorithm, which when presented with a solution candidate, gives us the probability that the candidate fits the equation. We call this a probabilistic or measure valued solution.

The earlier solution concepts are also probabilistic solutions. For instance, if we say that a solution of (1) is any fruit, we construct our probabilistic solution to report 100% if the input is a fruit, and 0% otherwise.

Mathematical models, in particular differential equations, are often used to study aspects of “real world” problems. In this context one wishes the model to have certain properties, and mathematicians sometimes have the following wish list:

1. There exists a solution.
2. There exists only one solution.
3. If we change the problem a little, the solution changes only a little.
4. The solution, or at least something close to it, can be calculated.

The first item on the list is understandable, since the model (question) will be meaningless otherwise. The second and third items are related, and if the model describes something in the real world, then its predictive value will be greatly reduced if 2 or 3 does not hold. Finally if 4 does not hold, then the model can not be used for precise quantitative statements. If all the items on the wish list hold, then we call the problem well posed.

Ordinary differential equations

Next, I shall try to indicate how such ideas are used when studying differential equations. Recall that the derivative of a function $x(t)$ evaluated at $t = s$ is the limit

$$\frac{dx}{dt}(s) = \lim_{h \to 0} \frac{x(s+h) - x(s)}{h}.$$

An ordinary differential equation is a relation of the type

$$\frac{dx}{dt} = f(t, x(t)). \tag{2}$$

where f is some known function, and we seek to determine the unknown function $x(t)$. The simple differential equation

$$\frac{dx}{dt} = 1,$$

has a solution given by

$$\tilde{x}(t) = t + c,$$

where c is any constant. In this case, and in the vast majority of cases, there is no ambiguity in how the equation (2) should be interpreted: For any (reasonable) t the functions on the left and right side in (2) exist and give the same number. Such solutions are often called classical solutions. There exists a general theorem saying that if f satisfies a few conditions and we are given $x(t_0) = x_0$, then there exists one and only one classical solution $x(t)$ to (2) such that $x(t_0) = x_0$. Furthermore, for t sufficiently close to t_0 we can calculate $x(t)$ to any precision.

Before we turn to partial differential equations, it is convenient to recall two basic results in differentiation. The *chain rule*

$$\frac{d(f(x(t)))}{dt} = \frac{df}{dx}\frac{dx}{dt},$$

and the *product rule*

$$\frac{d(f(t)g(t))}{dt} = \frac{df}{dt}g(t) + f(t)\frac{dg}{dt}.$$

It is also very useful to introduce the inverse of differentiation; *integration*. If $dx/dt(t) = y(t)$, and $x(\alpha) = 0$, then we write

$$x(t) = \int_{\alpha}^{t} y(\tau)\, d\tau,$$

and we call $x(t)$ the integral of y from α to t.

Partial differential equations

For functions of two variables, $u = u(x,t)$, we define the partial derivatives

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(x+h,t) - u(x,t)}{h} \quad \text{and} \quad \frac{\partial u}{\partial t} = \lim_{h \to 0} \frac{u(x,t+h) - u(x,t)}{h},$$

whenever these limits exist.

A partial differential equation is some equation involving an unknown function u and its partial derivatives. A simple example of a partial differential equation is

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{3}$$

We shall try to determine a function $u(x,t)$ such that this holds. Let $v(y)$ be any differentiable function (v is called the initial data, as t is commonly interpreted as time), and set

$$u(x,t) = v(x - t). \tag{4}$$

Set $y = x - t$, and use the chain rule to calculate

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = \frac{dv}{dy}\frac{\partial y}{\partial t} + \frac{dv}{dy}\frac{\partial y}{\partial x} = \frac{dv}{dy}(-1+1) = 0.$$

Thus we see that u is a solution. If v is differentiable, then u is called a classical solution. We also see that for a fixed t, the graph of $u(x,t)$ is the graph of $v(x)$ shifted t units to the right. This means that if v is differentiable, then we have existence of a classical solution to (3). Regarding uniqueness, assume that $\bar{u}$ is another classical solution with initial data $\bar{u}(x,0) = \bar{v}(x)$. Then the difference $u - \bar{u}$ is another weak solution with initial data $v - \bar{v}$ (here we use the linearity of (3)). Assume now that $u - \bar{u}$ is square integrable in x, i.e.,

$$\int_{-\infty}^{\infty} (u(x,t) - \bar{u}(x,t))^2 \, dx < \infty.$$

This is certainly the case if both u and $\bar{u}$ are square integrable in x. Now multiply the equation for $(u - \bar{u})$ by $(u - \bar{u})$ to get

$$\frac{\partial}{\partial t}\left((u-\bar{u})^2\right) + \frac{\partial}{\partial x}\left((u-\bar{u})^2\right) = 0.$$

Next, integrate this over x and find that

$$\frac{d}{dt}\int_{-\infty}^{\infty} (u(x,t) - \bar{u}(x,t))^2 \, dx = 0,$$

and hence that

$$\int_{-\infty}^{\infty} (u(x,t) - \bar{u}(x,t))^2 \, dx = \int_{-\infty}^{\infty} (v(x) - \bar{v}(x))^2 \, dx.$$

This means that we have both stability with respect to the initial data, and uniqueness of classical square integrable solutions. Furthermore, knowing the initial data, we can read the unique solution by the formula (4). Thus the entire wish list is fulfilled by classical solutions.

Even if v is not differentiable everywhere, the function defined by $u(x,t) = v(x-t)$ is a classical solution at those (x,t) such that v is differentiable at $x - t$. Therefore it is useful to introduce an extended solution concept, which makes the formula (4) valid even if v is not differentiable everywhere. To do this note that the integral formulation of the product rule is

$$f(t)g(t) = \int f(t)\frac{dg}{dt}\,dt + \int \frac{df}{dt}g(t)\,dt.$$

Now choose a so-called test function $\varphi(x,t)$, where φ is differentiable, and $\varphi(x,t) = 0$ if $|x| > M$ for some positive constant M. If u is a classical solution of (3), we can multiply (3) by φ, integrate over all x and for $t \in (0,T)$, and use the integral formulation of the product rule to find that

$$\int_0^T \int_{-\infty}^{\infty} \left(u(x,t)\frac{\partial \varphi}{\partial t} + u(x,t)\frac{\partial \varphi}{\partial x} \right) dx dt = \int_{-\infty}^{\infty} \left(u(x,T)\varphi(x,T) - u(x,0)\varphi(x,0)\right) dx. \tag{5}$$

Now we have transferred all the derivatives onto the test function φ, and we define a *weak solution* of (3) to be a function so that (5) holds for *all* test functions φ. All classical solutions are weak solutions, but the opposite does not hold. However, if the initial data is any function v such that the integral

$$\int_{\alpha}^{\beta} v(x)\,dx$$

is finite for all α and β, it is not difficult to show that $u(x,t) = v(x-t)$ is a weak solution. Thus we have existence of a weak solution if the initial data is locally integrable.

To show uniqueness, let $\bar{u}$ be another weak solution with initial data v. Set $w = u - \bar{u}$, then $w(x,0) = 0$. Now assume that the test function φ solves the equation

$$\begin{cases} \frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial x} = w(x,t), \; 0 \le t < T, \\ w(x,T) = 0. \end{cases} \tag{6}$$

Using this in the weak formulation for w, we get

$$\int_{-\infty}^{\infty}\int_{0}^{T} w^2(x,t)\,dxdt = 0,$$

and thus $w = 0$ or $\bar{u} = u$. Hence we have uniqueness for weak square integrable solutions.

Of course, it remains to show that the solution of (6) is an admissible test function. However, using the chain rule, one can show that φ is given by the formula (at this point we do not need uniqueness, only existence!),

$$\varphi(x,t) = -\int_{t}^{T} w(x-(t-s),s)\,ds.$$

With a few extra arguments, one then shows that this can be used in the weak formulation (5).

To show stability we observe that if $\bar{u}$ solves the equation with initial data $\bar{v} \neq v$, then w will solve the equation with initial data $v - \bar{v}$. Now we know that this problem has the unique solution $v(x-t) - \bar{v}(x-t)$. Therefore a small change in the initial data will result in a small change in the solution. So, also in the class of weak solutions, the initial value problem for (3) is well posed.

Conservation laws

Conservation laws are a nonlinear generalization of (3). In one dimension a conservation law reads

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0. \tag{7}$$

The interpretation of this equation is that the total amount of u is conserved, with a flux density given by $f(u)$. Therefore, conservation laws are common in models from physics, in particular fluid dynamics, where the conserved quantities are often mass, momentum and energy. In this case the conserved variable is a vector, but scalar conservation laws serve as an important example.

The prototype of a scalar conservation law is Burgers' equation, where $f(u) = u^2/2$. If u is a classical solution to Burgers' equation, then by the chain rule, u also satisfies

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

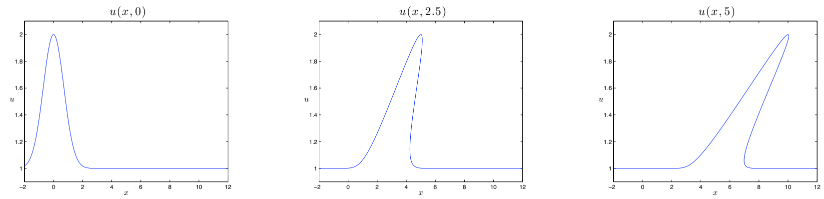


Figure 1. Burgers' equation, a breaking wave.

Now let $v(x)$ be any differentiable function, and let $y(x,t)$ solve

$$x - tv(y(x,t)) = y(x,t).$$

Observe that $y(x,0) = x$. Using both the product and the chain rule we find

$$\frac{\partial y}{\partial t} = \frac{-v(y)}{1+t\frac{dv}{dy}(y)}, \ \frac{\partial y}{\partial x} = \frac{1}{1+t\frac{dv}{dy}(y)}.$$

Now define $u(x,t) = v(y(x,t))$ and compute

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= \frac{dv}{dy}\frac{\partial y}{\partial t} + v\frac{dv}{dy}\frac{\partial y}{\partial x} \\ &= \frac{\frac{dv}{dy}}{1+t\frac{dv}{dy}}(-v+v) = 0.\end{aligned}$$

Hence u is a classical solution with the initial data $v(x)$. However, if $dv/dy(x) < 0$ for some x, then $1 + t\,dv/dy(x) = 0$ for some $t > 0$. At this point the partial derivatives of y "blow up", and the derivatives of u are not defined. Thus a classical solution cannot exist for all positive t! Therefore, weak solutions seem the appropriate solution concept in this case. For conservation laws weak solutions are defined by equation (5), but with $f(u)\partial\varphi/\partial x$ replacing of $u\partial\varphi/\partial x$.

For Burgers' equation the speed of transport depends on the value of what is transported, so that the top of the a wave will move faster than the bottom, and the wave will break. This looks similar to water waves breaking near a beach. Figure 1 shows an example of this.

The problem is that the solution becomes "multivalued" as the wave breaks, and we need to remove some parts of the graph where we have multiple values. This must be done carefully, so that the result is still a weak solution. However, there are many ways to do this, and it turns out that one needs an additional principle to single out a unique weak solution.

This is usually done by adding a so called "entropy condition", stating that information may be lost, but never created. The entropy condition gives time, or the t variable, a direction. Hence, in sharp contrast to (3), the conservation law (7) becomes time irreversible after the addition of an entropy condition.

With the addition of the entropy condition, the initial value problem for scalar conservation laws is well posed.

However, there is no general closed formula for the entropy weak solution to the initial value problem. Therefore, one must use numerical methods to compute approximations. These approximations are usually based on replacing the derivatives in the equation by finite differences, i.e., by stopping short of the limit in the definition of the derivative. However, one needs to be very careful in how this is done,

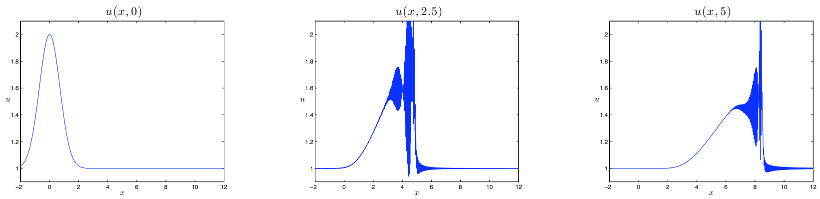
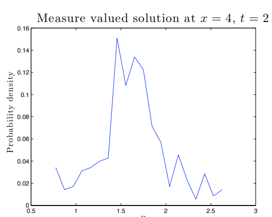


Figure 2. High order approximation to a breaking wave.

since we know that the entropy solution is usually discontinuous, and certainly not differentiable.

If this careful approximation of the derivatives is *low order*, then the resulting function is guaranteed to be close to the entropy solution. But low order means that one has to use large computational resources to get close to the solution. Therefore, for practical computations, higher order methods are much more common, but for these approximations we do not have the guaranteed convergence. Figure 2 shows the result of a high order approximation to the entropy solution from Figure 1. Formally, the solution in Figure 2 should be very close to the entropy solution, yet the approximation is so oscillatory that it is difficult to say what we are looking at. From theoretical considerations, we know that the entropy solution is no more oscillatory than the initial data, so the graph of the approximation seems meaningless.



Nevertheless, one can actually show that the high order approximation is close to yet a third type of solution, a *measure valued solution*. This is a procedure, which for each point (x,t) and for each interval (α,β), gives the probability that the solution at the point (x,t) is in the interval (α,β). In addition, one can show that there exists at a unique measure valued solution which also satisfies an entropy condition and takes the initial data in a suitable sense. Thus, if the initial data are such that a unique entropy weak solution exists, then this coincides with the entropy measure valued solution. The figure to the left shows an approximation of the probability density of the measure valued solution at $(x,t) = (4,2)$.

To anyone interested in partial differential equations, in particular conservation laws, and their solutions, I recommend the following books as a starting point:

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