HANS L. PÉCSELI

Limits to Predictability and Understanding, seen from a Physicist's Perspective

As an introduction, we may consider an infinitely many times differentiable function F of some variable ξ. Presuming we have complete information of this function in a narrow interval at a reference ξ_0, we can predict the values of this function at any $\xi > \xi_0$ by a MacLaurin series expansion

$$F(\xi) = F(\xi_0) + \frac{F^{(1)}(\xi)}{1!}(\xi - \xi_0) + \frac{F^{(2)}(\xi)}{2!}(\xi - \xi_0)^2 + \ldots + \frac{F^{(n)}(\xi)}{n!}(\xi - \xi_0)^n + \ldots$$

Professor Hans L. Pécseli
Department of Physics,
University of Oslo, Norway
hans.pecseli@fys.uio.no
CAS Fellow 2004/2005

where we by $F^{(n)}(\xi_0)$ understand the n-th derivative, $d^nF(\xi)/d\xi^n$, taken at the position ξ_0. It can thus be argued that given the a-priori knowledge of infinitely many times differentiable functions, we have complete knowledge of F for all ξ, provided we have been able to obtain the knowledge of all $F^{(n)}$ in just one position ξ_0. If we take ξ to represent a time variable t, and $F(t)$ to be the output signal from a signal transmission line, we can then argue that the signal does not contain any information for $t > t_0$ since we have complete knowledge of it already at $t = t_0$. In this sense, information is associated with discontinuities in either the signal $F(t)$, or some of its derivatives, and indeed most methods of signal transmission use pulsed signals of various forms.

The use of the MacLaurin series is, unfortunately, not quite as simple as it may seem: in the vicinity of a reference variable we may in reality be given the function either as a long, but finite, list of numbers or alternatively in the form of a graphical representation. Assume that in the former case we have, say, N points at our disposition: we are then able to estimate N-1 derivatives, i.e. up to $F^{(N-1)}$, unfortunately with an error increasing with the order of the derivative, in part also because we will only have a finite number of digits available, and the numerical differentiations become uncertain also because of round-off errors. If the function is given graphically as a curve in an interval, the finite width of the line and other practical problems make the derivative estimates even more uncertain. In *reality*, we will benefit from the MacLaurin series only for some time, and when the errors due to our uncertainty on $F^{(n)}$ and ignorance of higher derivatives accumulate, our predictions will become increasingly uncertain and ultimately simply wrong, in spite of the formal possibility of ideal predictability. In addition we might of course encounter cases where the function may not be differentiable at a possibly infinite set of ξ-values, in which case our efforts to achieve long time predictions would be in vain anyhow. Such cases have practical importance as already mentioned.

The discussion outlined here made reference to one variable only, but applies equally well for functions of many variables, in particular also for cases where the independent variables are time and spatial positions, i.e. $F = F(x, y, z, t)$. A relevant case could be where F represents the space-time varying smoke concentration emerging from a smokestack which is transported and mixed by the turbulent flow in the atmosphere (e.g. Tennekes & Lumley 1972). Turbulence in nature as well as laboratory experiments represents great challenges for our understanding of phenomena with many degrees of freedom, but in addition it is of great practical importance for pollution transport in the environment. It is important also for certain aspects of the biological food chain as discussed in a different context. It is safe to argue that turbulence is of great importance for everyday life, and significant efforts have been made to understand and describe turbulent flows.

When considering turbulent fluids, as visualized for instance by a smoke plume from a smokestack, it seems inconceivable that such conditions are amenable for analytical studies. Nonetheless it was demonstrated first by Kolmogorov that given some reasonable simplifying assumptions, certain statistical averages can be predicted to a high degree of accuracy, such as the second order structure function, which is defined as the mean square of the velocity difference between two positions in the fluid. The idea was in a sense familiar to physicists already: for example the kinetic theory of gases is a study of statistical averages of measurable quantities associated with many interacting particles. In the early days of classical statistical mechanics, scientists were confident that, at least in principle, any problem involving many interacting particles could be solved in a certain sense, and statistical mechanics was at least to some extent seen as a means of reducing the relevant information to a manageable level. In our imagination, we might consider a situation where a very large number of different springs carrying particles with different masses and all of them interconnected. The problem could in principle be analyzed completely within classical mechanics, but any attempt to do so would be made impossible by the immense number of degrees of freedom that has to be accounted for in the analysis. In particularly simple cases we are actually able to provide an exact solution even for realistic cases, but even if this were possible for a general case, such a result would contain too much information to be manageable in practice.

Classical turbulence models are to some extent related to these problems from statistical mechanics: in models of fluid turbulence we consider the medium as a continuum and ignore its particle properties, but nonetheless turbulence is understood and described as a physical condition where many structures interact, see Figure 1. The basic description in terms of a partial differential equation of incompressible fluids is well established (the Navier-Stokes equation), but its solution for general flows is close to impossible, and even if such a solution were available, it would be of little practical use by containing much more information than is manageable for an actual application. The Navier-Stokes equation is stated as,

$$\frac{\partial}{\partial t}\mathbf{u}(\mathbf{r},t)+\mathbf{u}(\mathbf{r},t)\cdot\nabla\mathbf{u}(\mathbf{r},t)=-\frac{1}{\rho}\nabla p+\nu\,\nabla^2\mathbf{u}(\mathbf{r},t)$$

where $\mathbf{u}(\mathbf{r},t)$ is the space-time varying fluid velocity vector, p is the pressure, ρ is the mass density of the fluid (usually assumed to be a constant) and ν is the kinematic viscosity, which accounts for the energy dissipation in the medium as heat. Viscosity is effective predominantly for small scales. (It is important to note that the mathematical structure of the equation becomes significantly different if dissipation as given by the last term in the Navier-Stokes equation is omitted, since the equation then only contains first order spatial derivatives.) The equation is completed by an expression for incompressibility, $\nabla\cdot\mathbf{u}(\mathbf{r},t)= 0$, which is appropriate for motions with small Mach numbers, i.e. much slower than the velocity of sound. Many relevant turbulence conditions in the oceans or in the atmosphere fulfill this condition. As outlined in the introduction, the idea of predictability of function (here the space-time varying velocity vector field) requires that the function is infinitely many times differentiable at all spatial positions at all times. We can of course ensure that the initial condition in a computer simulation, for instance, has this property, but there is no a-priori guarantee that such singularities can not develop spontaneously at later times. The problem is not completely resolved, but there seems to be a consensus that a statistically homogeneous and isotropic turbulent velocity field remains infinitely many times differentiable as long as the Navier-Stokes equation applies as a model. The Navier-Stokes equation is mathematically relatively simple, but has very complicated solutions, which only recently have been explored numerically by powerful computers, and even then only for cases limited to weakly developed turbulence.

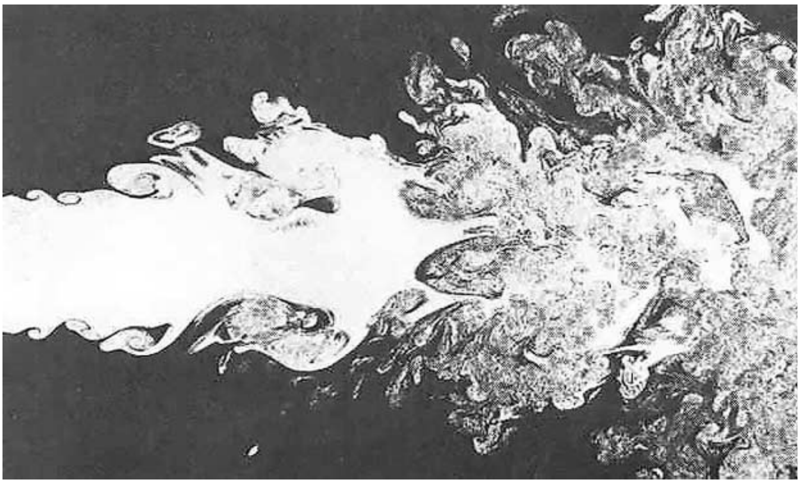

Figure 1. The figure shows a "jet" of high velocity being injected from the left side into a background fluid at rest. Note the evolutions of small coherent structures, which subsequently mix into a turbulent flow to the right of the figure.

The standard and mathematically strict way of describing structures in turbulence is by means of a Fourier transform of the velocity field, but for the present purpose some qualitative ideas will suffice. The most important point is the basic statement that under statistically steady state conditions, turbulence can be seen as a competition of energy input and energy dissipation. Energy is injected into the system by some external force, in laboratory experiments for instance by rotating propellers, moving grids or similar. The energy is assumed to be injected at large scales, for most relevant cases. With the scales interacting, the energy will be fed on to smaller and smaller scales, until it is ultimately dissipated at the smallest scales by the fluid viscosity. In turbulence studies the notion of "energy cascade" is invoked for the process. If we arrest the energy input, the turbulence will decay, and the motions in the fluid will eventually crank to a halt, the largest scales surviving the longest.

The visual impression of turbulent flows is associated with chaos, implying unpredictability. It could be tempting to assume that chaos was uniquely associated with systems having many degrees of freedom. One important observation which dates back to the 19th Century is that chaos, in the sense of unpredictability, can be found also in systems with few degrees of freedom: it can be demonstrated that three suffice (e.g. Schuster 1988). As strange as it may sound, it can be demonstrated that, for instance, three planets interacting by gravitational forces can exhibit regular motion for extended periods of time, until one of them suddenly tears loose and can leave the system. Any prediction made on past observations suggesting regular motion for all future times will be in error. For systems like these, chaos is manifested by lacking predictability. We can observe and describe the system at any instant, but we can only do so with a finite accuracy. If we compare two initial conditions which are both compatible within a certain accuracy, the system can develop into very different later states. Even the slightest error or inaccuracy of our description of a state of the system at a given reference time can result in a prediction of future states which is in grave error, even though we purport to know the basic laws of motion exactly. Unfortunately, meteorological models for weather prediction have such properties: even the simplest reasonable model we might imagine, consisting of only three coupled partial differential equations (related to the Lorenz model), has such a property. The term "butterfly effect" has been coined to describe the great uncertainty in the predictions that we as laymen have observed so often when comparing weather forecasts with actual observations. "The flapping of the wings of a butterfly can give rise to perturbations in our predictions and give results which are completely different from what we find when these perturbation are absent." If we attempt to make weather predictions using a reasonably accurate numerical model, our results will be extremely sensitive to the input data, sensitive even to the "flapping of the wings of a butterfly". If we, for instance, specify our initial conditions to, say, 8 digits accuracy, we will find a substantial difference in the predicted evolution if we change the last digit. For a while, the two solutions are likely to follow each other, but the difference will slowly increase, eventually becoming very large. The time it takes for substantial deviations to develop will be one of the characteristics of the problem. In spite of this chaotic feature, the basic model equations will still ensure that identical conditions lead to identical results.

The basic idea of making predictions and models for average quantities in many-body systems has been widely used in natural sciences. The well known Ohm's law, relating electric currents to potential differences is one example, and the diffusion equation or the slightly simpler Fick's law, are examples of cases where such a model has proved to be valuable. These laws describe in a simple way immensely complicated underlying physical phenomena. Their success relies on the fluctuations around these averages being small. Heuristic model equations are often used also in social sciences for predicting the behaviour of societies, which after all can also be seen as systems of many interacting bodies. The limitations of such models have been made conspicuous by noting that many of these model equations have chaotic solutions.

It seems appropriate here to add some comments relating to "modern physics" or quantum mechanics, which also deals with uncertainties of predictions, albeit, as argued in the following, by completely different reasoning. First of all, it might be stated quite trivially, that uncertainties in measurements were, of course, recognized in classical physics. The uncertainties dealt with in quantum mechanics are concerned with limitations in our basic understanding, and should be seen as something quite different. It might be appropriate first to insert a few ideas concerning the meaning of the word "understanding". Here it should be emphasized that the discussion will be somewhat restrictive: we might claim to understand a piece of music or a painting, for instance, but these aspects of the word will not be covered here. Rather, we might start with the first human encounter with the word "understanding": a child experiencing the first causal relationships. Assume that it knocks over an ugly, old vase you hated anyway, so you take it in quite a relaxed way with a few words like "never mind, it was not really your fault, etc…". Next time, however, it is a precious gift that goes, and this time your reaction becomes much more angry. The child sees no difference in its actions (and indeed with some right), but the consequences are dramatically different. The poor child becomes all confused, it *does not understand*: the causal relations (and the related social relations) it tries to build get all messed up. When seen in this light, we might argue that we *understand* a phenomenon, when we are able to predict the consequences of a given initial condition, or alternatively, given an effect, we can be certain of its cause. The virtue of this formulation is that "understanding" in the present sense can be communicated, and understanding enters also as an element in social or in human relations. We can explain a cause by words or other ways and predict its consequences for somebody else. Many other experiences, no matter how emotionally deep or profound, may remain "private" because we lack words or other means of communicating this experience.

Within classical mechanics, or rather classical physics in general, the possibility of a complete understanding in the sense illustrated before seemed within reach around the year 1900, or so. For sure, scientists were fully aware that any cause, or its effect, could be measured and determined with a finite accuracy only. However, physicists had for centuries seen the accuracy of their measurements increase almost unbelievably. On the distant horizon a nearly perfect accuracy could be imagined, one which allowed measurement of a given state of matter to arbitrary high (though finite) precision, with a subsequent prediction of future states with correspondingly high precision: what we could claim to be a *complete understanding* of nature. The emergence of quantum mechanics reduced this ideal to a shambles (e.g. Davydov 1969). As may best be illustrated by the model experiment known as "Heisenberg's microscope", we learned that a state of matter, let this be constituted by just a single electron, can be measured with finite accuracy only. Basic laws of physics set the limit to this accuracy as expressed in terms of Planck's constant, and not human limitations in experimental accuracy. We might, for instance, claim to know the position of a particle to any precision, but only at the expense of a corresponding ignorance on its velocity. The product of the two uncertainties is larger than a certain minimum value, the relation known as *Heisenberg's uncertainty principle*, stated here as

$$\Delta x \, \Delta p > h \, ,$$

where Δx is the uncertainty in the determination of the position of a particle (here for a one dimensional model) and Δp is the corresponding uncertainty in the particle momentum (*mass* × *velocity*), while h is Planck's constant, where we in classical mechanics would set Planck's constant to zero, thus allowing, at least in principle, $\Delta x = 0$ and $\Delta p = 0$ simultaneously. The consequences of the uncertainty relation for our ideas of "understanding" are profound: we will never be able to define a causal relationship accurately, since the input data will always be undetermined by the uncertainty principle, and consequently we will never be able to reach a complete understanding of a physical problem, at least not in the sense of the word as argued above. There is a profound epistemological difference between limitations of predictability caused by fundamental laws of nature, and those caused by practical problems like round-off errors. These realizations lead to a complete reformulation of physics, known as quantum mechanics. This is still an exact science, but its predictions do not concern individual states of matter or events but rather probabilities of states, where the probability density can now be predicted with accuracy. These predicted probability densities can be estimated experimentally, but no longer on the basis of a single set of observations obtained from just one realization of the experiment.

Acknowledgements

The author would like to thank Jan Trulsen and Jonathan Williams for valuable discussions.

References

Davydov, A. S. (1969) *Quantum Mechanics*. Pergamon Press.

Schuster, H. G. (1988) *Deterministic Chaos, an Introduction*. 2nd ed. VCH Verlagsgesellschaft.

Tennekes, H. & Lumley, J. L. (1972) *A First Course in Turbulence*. MIT press.