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Computing Approximate Centroids in a Polyhedron

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Abstract. We discuss about efficient algorithms for obtaining the centroid direction for each of the three types of monotonicity in a polyhedron. Strongly- and directionally-monotone centroids are shown to be obtained by applying the previous result. This paper focuses on developing an efficient method for approximating the weakly-monotone centroid.

Keywords: Polyhedron Monotonicity, Centroid Direction, Spherical Algorithm

1 Introduction

Three types of a polyhedron monotonicity: *strong*, *weak*, and *directional monotonicity* have been characterized as geometric problems to find great circles *separating or intersecting* a set of spherical polygons that are derived from sub-surfaces of the polyhedron and its convex hull [1]. Consequently, all directions for the three monotonicities can be constructed in $O(nk \log k + n \log n)$ time, where n and k are the numbers of all faces and all sub-surfaces, respectively.

In this paper, we consider efficient algorithms for finding a centroid of all monotone directions, which will be called the *monotone centroid* in short, in a polyhedron. The centroid in a set of directions is defined so as to maximize the minimum distance between the centroid and all directions in the set.

The strongly- and directionally-monotone centroids can be obtained by directly applying a discrete algorithm [2] that approximates centroids among great circles maximally intersecting a set of spherical polygons. The weakly-monotone centroid will be efficiently approximated in $O(n)$ time by intersecting other spherical objects called *great bands* instead of non-convex spherical regions.

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2 Notations and Definitions

The space on the boundary of the unit sphere centered at origin in three dimension is described as $S^2 = \{p \mid \|p\| = 1\}$. A point p on S^2 is a unit vector in 3 dimensional Euclidean space E^3.

A circle on S^2 is determined by the intersection of the unit sphere with a plane. If the plane contains the origin, the intersection is called a *great circle*; otherwise, it is called a *small circle*. The circle is denoted by $Cr(p,\theta) = \{x \mid p \bullet x = \cos\theta\}$. We call p the *pole* of the circle, and θ the size of the circle.

The *great band* bounded by two circle on S^2 is denoted by $GB(p,\theta_u,\theta_l) = \{x \mid \cos\theta_l \le p \bullet x \le \cos\theta_u\}$, where $\theta_l > \frac{\pi}{2}$ and $\theta_u < \frac{\pi}{2}$. The 3D space bounded by planes passing the two circles with $Sp(p,\theta_u,\theta_l) = \{x \mid \cos\theta_l \le p \bullet x \le \cos\theta_u, x \in E^3\}$,

The set $U = \{u_1,\ldots u_n\}$ of outward unit normal vectors of a surface F is called the *Gaussian map* of F. The spherical convex hulls of the Gaussian map of F will be denoted by $GCH(F)$. The *visibility map* of F is the set of directions visible to F.

3 Approximating Weakly-Monotone Centroids in a Polyhedron

Monotone directions of a polyhedron can be characterized with the sub-surfaces of the polyhedron: pockets, lids, sub-pockets, and sub-lids. Weakly-monotone directions can be established by *finding great circles intersecting a set of visibility polygons* of sub-pockets and sub-lids of a polyhedron (Lemma 6 in [1]). Intersecting a set of spherical polygons is *complementary* to separating the set of spherical polygons. The poles of great circles separating a visibility polygon are the complement of its positive and negative duals.

We introduce two circles bounding a convex polygon; *in-circle* and *circum-circle*, which are the largest circle within the polygon and the smallest circle enclosing the polygon, respectively. When we replace a polygon with its bounding circle, the non-convex region that is the complement of two polygons is approximated with a convex object called the *great band*, as illustrated in Figure 2. The in-circle approximation can be used for the feasibility test since it is a *necessary* condition for the original solutions, while the circum-circle is used for a *sufficient* approximation such as finding the centroid among solutions.

The approximately reduced problem of intersecting great bands is considered under the extension of the geometric space from S^2 space into 3D space. It is obvious that the intersection of a set of bands on S^2 is empty ($\bigcap GB(p_i,\theta_{u_i},\theta_{l_i}) = \phi$), if and only if the closed convex polyhedron yielded by $\bigcap Sp(p_i,\theta_{u_i},\theta_{l_i})$ is completely enclosed by S^2. In order to check without constructing all boundary of $\bigcap Sp(p_i,\theta_{u_i},\theta_{l_i})$, we find its extreme point with a maximizing problem: maximize $\|x\|^2$ subject to all $\{Sp(p_i, \theta_{u_i}, \theta_{l_i})\}$. After determining the extreme point x^* of $\bigcap Sp(p_i, \theta_{u_i}, \theta_{l_i})$, a simple test $\|x^*\|^2 < 1$ is performed.

The centroid of $\bigcap GB(p_i, \theta_{u_i}, \theta_{l_i})$ is the center of circle inscribing the solution region of $\bigcap GB(p_i, \theta_{u_i}, \theta_{l_i})$, the boundary of which is composed of the parts of small circles $Cr(p_i or -p_i, \theta_{u_i} or \theta_{l_i})$. In other words, a weakly-monotone centroid can be obtained by computing the center of an inscribing circle. We can get a reduced problem for weakly-monotone directions in a polyhedron, which can be solved in $O(n)$ time by linear programming [3], as the following Lemma.

Lemma 1 The unit vector of a solution x^* maximizing $\|x\|^2$ subject to all $\{Sp(p_i, \theta_{u_i}, \theta_{l_i})\}$ is the centroid of $\bigcap GB(p_i, \theta_{u_i}, \theta_{l_i})$.

The next discussion is how to construct the two bounding circles: in-circle and circum-circle. The circum-circle that is sometimes called *the smallest enclosing circle* in 2D can be constructed with a simpler formulation in an efficient $O(n)$ time [4]. The circum-circle bounding a spherical polygon on S^2 can be obtained by constructing *the smallest sphere* enclosing a set of points in 3D as the following Lemma.

Lemma 2 The intersection of S^2 and the smallest sphere enclosing the vertices of a polygon is the circum-circle of the polygon.

Finding the three edges of a convex polygon in 3D for its in-circle is a combinatorial problem up to ${}_nC_3$ circles may be tangential to three of n edges in the polygon. Even though there is an optimal $\theta(n \log n)$ algorithm [5] for this problem, we can get the in-circle in $O(n)$ on S^2 by using the surprising result [6]; the in-circle of a polygon is complement of its circum-circle on S^2, where two circles $Cr(p_1, \theta_1)$ and $Cr(p_2, \theta_2)$ on S^2 are said to be *complementary* to each other if $p_1 = p_2$ and $\theta_1 + \theta_2 = \pi / 2$.

4 Results

By using Lemma 1 and 2, we can construct an efficient algorithm for obtaining the weakly-monotone centroid with the sub-pockets and sub-lids of a polyhedron can be constructed as.

procedure WeaklyMonotoneCentroid
input: the sub-pockets $\{SP_i\}$ and the sub-lids $\{SL_j\}$ of P

output: the weakly-monotone centroid of P

step 1. For each i, j, compute $\{GCH_n(SP_i \cup SLF_{jk})\}$,

where SLF_{jk} is each face $\in SL_j$.

step 2. For each n, determine the smallest $Cr(p_n, \theta_n^C)$ enclosing GCH_n [4,7,8].

step 3. Find the extreme x^I of $\bigcap Sp(p_n, \theta_n^I, \pi - \theta_n^I)$ with LP [3,9,10],

where $\theta_n^I = \pi / 2 - \theta_n^C$.

step 4. If $\|x^I\|^2 < 1$ then, terminate with the result of infeasibility.

step 5. Find the extreme x^C of $\bigcap Sp(p_n, \theta_n^C, \pi - \theta_n^C)$ with LP [3,9,10].

endProcedure WeaklyMonotoneCentroid

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References

1. J.S. Ha and K.H. Yoo.: Characterization of polyhedron monotonicity. Computer-Aided Design, vol. 38, no. 1, pp. 48-54 (2006).
2. J.S. Ha and K.H. Yoo.: Approximating centroids for the maximum intersection of spherical polygons. Computer-Aided Design, vol. 37, no. 8, pp. 783-790 (2005).
3. M.E. Dyer.: Linear time algorithms for two- and three-variable linear programs. SIAM. Journal on Computing, vol. 13, no. 1, pp. 31-45 (1984).
4. E. Welzel.: Smallest enclosing disks (balls and ellipsoids). New Results and New Trends in Computer Science, Springer Lecture Notes in Computer Science, vol. 555, pp. 359-370 (1991).
5. G.T. Toussaint.: Computing largest empty circles with location constraints. International Journal of Computer and Information Sciences, vol. 12, no. 5, pp. 347-358 (1983).
6. J.G. Gan, T.C. Woo and K. Tang.: Spherical maps: their construction, properties, and approximation. ASME J. Mechanical Design, vol. 116, pp. 357-363 (1994).
7. N. Capens's implementation of [6], http://www.flipcode.com/archives/Smallest_Enclosing _Spheres.shtml.
8. CGAL implementation, http://www.cgal.org/Manual/latest/doc_html/cgal_manual/ Bounding_volumes_ref/ Class_Min_sphere_of_spheres_d.html.
9. M. Hohmeyer's implementation of [9], ftp://icemcfd.com/pub/linprog.a.
10. CGAL implementation, http://www.cgal.org/Manual/latest/doc_html/cgal_manual/QP _solver/Chapter _main.html.

