A Study on Discrete WALSH Transforms for Digital Control Systems Analysis

Joon-Hoon Park[1], Ryum-Duck Oh[2]

[1] Department of Control and Instrumentation Engineering, Korea National University of Transportation,
Chungbuk, Korea. jhpark@ut.ac.kr
[2] Department of Software, Korea National University of Transportation,
Chungbuk, Korea. rdoh@ut.ac.kr

Abstract. Walsh functions were completed from the incomplete orthogonal function of Rademacher in 1923. Walsh functions form the complete orthonormal set of rectangular types and have three classified groups. These groups differ from one another in that the order in which individual functions appear is different. Walsh functions and its discrete transforms have useful analog-digital properties and are currently being used in a variety of engineering applications, which include image processing, digital filtering, simulation, signal processing and control systems. In this paper discrete Walsh transforms is suggested for digital control systems analysis.

Keywords: Walsh functions**,** discrete transforms, digital control systems, system analysis

1 Introduction

The Rademacher functions are a set of square waves for $t\in[0, 1)$, of unit height and repetition rate equal to 2^m, which can be generated by a BCD counter. The Walsh functions constitute a complete set of two values orthonormal functions $\Phi_k(t)$, $k=0, 1, 2, ...n-1$, $n=2^m$ in the interval (0, 1), and they can be defined in the several equivalent ways.

The set of Walsh functions is generally classified into three groups. The three type of Walsh orderings are a) Walsh ordering, b) Paley ordering and c) Hadamard ordering. First, Walsh ordering is originally employed by Walsh. We can denote Walsh functions belonging to this set by

$$S_w=\{Wal_w(i, t),\ i=0, 1, ..., N-1\} \qquad (1.1)$$

where $N=2^n$, $n=1, 2, 3.....$
The subscript w means Walsh ordering, and i denotes the i-*th* member of S_w. The *cal* and *sal* functions corresponding to $wal_w(i, t)$ are denotes as

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$Cal(s_i, t)=Wal_w(i, t)$, i even
$Sal(s_i, t)=Wal_w(i, t)$, i odd (1.2)

Second, the Paley ordering is dyadic type functions. Walsh functions are elements of the dyadic group and can be ordered using the Gray code. This Paley ordering of Walsh functions is denoted as

$S_p=\{Wal_p(i, t), i=0, 1, ..., N-1\}$
$Wal_p(i, t)= Wal_w (i_g, t)$ (1.3)

where i_g is the Gray code to binary conversion. The subscript p means Paley ordering. Third, Hadamard ordering can be denoted by

$S_h=\{Wal_h(i, t), i=0, 1, ..., N-1\}$
$Wal_h(i, t)= Wal_w (i_b, t)$ (1.4)

where i_b is the bit reversal of i. The subscript h means Hadamard ordering.
For the purpose of illustration, Table 1 is the results of evaluation for $N=8$ and the table shows relationship between the Walsh ordering and Hadamard ordering Walsh functions.

Table 1. Relationship between Walsh, Paley and Hadamard ordering

i	Paley to Walsh Odering	Hadamard to Walsh Odering
0	$Wal_p(0, t)=Wal_w(0, t)$	$Wal_h(0, t)=Wal_w(0, t)$
1	$Wal_p(1, t)=Wal_w(1, t)$	$Wal_h(1, t)=Wal_w(7, t)$
2	$Wal_p(2, t)=Wal_w(3, t)$	$Wal_h(2, t)=Wal_w(3, t)$
3	$Wal_p(3, t)=Wal_w(2, t)$	$Wal_h(3, t)=Wal_w(4, t)$
4	$Wal_p(4, t)=Wal_w(7, t)$	$Wal_h(4, t)=Wal_w(1, t)$
5	$Wal_p(5, t)=Wal_w(6, t)$	$Wal_h(5, t)=Wal_w(6, t)$
6	$Wal_p(6, t)=Wal_w(4, t)$	$Wal_h(6, t)=Wal_w(2, t)$
7	$Wal_p(7, t)=Wal_w(5, t)$	$Wal_h(7, t)=Wal_w(5, t)$

2 Discrete Walsh Functions

One of the earliest works in which discrete orthogonal transforms including Walsh functions were applied to the analysis processing of digital control systems and speech signals. And interest has grown in the possibility of using orthogonal transforms as a means of reducing the bit rate necessary. By sampling the Walsh functions shown in figure 1, we can obtain the *(8x8)* matrix shown in figure 2. In general an *(NxN)* matrix would be obtained. We denote such matrices by $H_w(n)$, since they can be obtained by reordering the row of a class of matrices called Hadamard matrices.

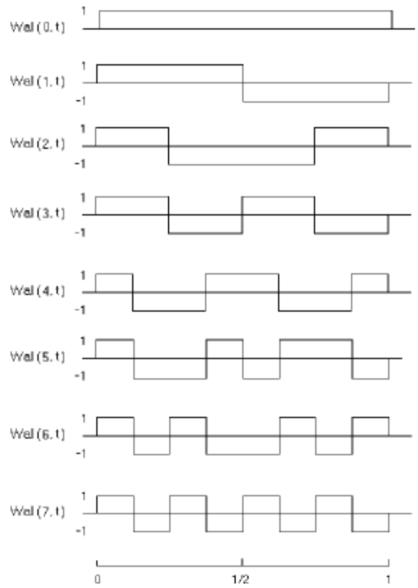

Fig. 1. Walsh ordering continuous Walsh functions, *N=8*

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Fig. 2. Walsh ordering discrete Walsh functions, *N=8*

Let u_i and v_i denote the *i-th* bit in the binary representations of the integers *u* and *v* respectively;

$(u)_{decimal}=(u_{n-1}\ u_{n-2} \cdots u_1\ u_0)_{binary}$ (2.1)

$(v)_{decimal}=(v_{n-1}\ v_{n-2} \cdots v_1\ v_0)_{binary}$ (2.2)

Then the elements ( ) of $H_w(n)$ can be generated as follows;

( ) ( )Σ7/ ( ) (2.3)

where $r_0(u)=u_{n-1}$,

$r_1(u)=u_{n-1}+u_{n-2}$,

$r_2(u)=u_{n-2}+u_{n-3}$, ….., $r_{n-1}(u)=u_1+u_0$

3 Discrete Walsh Transforms

The discrete Walsh transforms has found applications in many areas, including signal processing, pattern recognition and digital control systems. Every function $f(t)$ which is integrable is capable of being represented by Walsh series defined over the open interval (0, 1) as

$$x(t)=a_0+a_1 Wal(1,t)+a_2 Wal(2,t)+\ldots\ldots \qquad (3.1)$$

where coefficients are given by

$$\int \; (\;) \quad (\;) \qquad (3.2)$$

From this we are able to define a transform pair

$$(\;) \Sigma \quad (\;) \quad (\;) \qquad (3.3)$$

$$(\;) \int \; (\;) \quad (\quad) \qquad (3.4)$$

The integration shown in equation (3.4) may then be replaced by summation, using the trapezium rule on N sampling points, x_i, and we can write the finite discrete Walsh transform pair as

$$(3.5)$$

$$\Sigma \qquad (\;) \qquad (3.6)$$

Similar transforms, $X_c(k)$ and $X_s(k)$ can be obtained for a time series, x_i using Harmuth's *Cal* and *Sal* functions

$$(\;) \quad \Sigma \qquad (\;) \qquad (3.7)$$

$$(\;) \quad \Sigma \qquad (\;) \qquad (3.8)$$

Let f_n^* denotes sampling of $f(t)$

$$\Sigma \qquad (\;) \qquad (3.9)$$

And *i-th* discrete coefficients are given by

$$\Sigma \qquad (\;) \qquad (3.10)$$

From equation (3.9) and (3.10), we can write f_n^* as follows:

$\int$ ( )

( ) ( ) (3.11)

(3.12)

(3.13)

4 Examples

We assume that *f(t)=t, [0, 1)* and let us find the discrete coefficients of Walsh transforms *F*.

$\int$ ( ) (4) (4) (4.1)

Then we can define f_n^* and F_n as follow;

$f^*_0=0.125, f^*_1=0.375, f^*_2=0.625, f^*_3=0.875$ (4. 2)

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Table 2. Walsh Transform coefficients for a simple sine waveform, *N=32*

0.000	**0.663**	**0.063**	**0.000**	**0.000**	**-0.263**	**0.025**	**0.000**
0.000	-0.052	-0.006	0.000	0.000	-0.126	0.013	0.000
0.000	-0.013	-0.002	0.000	0.000	0.006	0.000	0.000
0.000	-.0.025	-0.002	0.000	0.000	-0.062	0.006	0.000

5 Conclusion

The analysis of digital control systems via discrete Walsh transforms is presented in this paper. The properties of continuous and discrete Walsh transforms are studied also. Aspects such as Walsh transforms expression in term of function derivatives, relation of ordering type with Walsh functions, and relation arithmetic and logical processing are considered. In digital control system analysis, application of discrete Walsh transforms is an useful method because of its reduced calculation burden and relative accuracy. In fact the wide applications of Walsh functions and transforms in control and signal processing research fields should be of interest.

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