Outer Synchronization of Complex Networks with Multiple Coupling Time-varying Delays*

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Abstract

This paper addresses the outer synchronization problem of complex networks with multiple coupling time-varying delays. First of all, some novel synchronization criteria are obtained by employing the Lyapunov stability theory and linear matrix inequality (LMI), we design some synchronization controllers to keep the given complex network synchronizing to the object trajectory. Furthermore, an adaptive outer synchronization scheme is derived to achieve global synchronization, which is simpler than some traditional controllers. Moreover, the presented results here can also be applied to complex networks with single time delay case. Finally, numerical analysis and simulations for two coupled complex networks which are composed of unified chaotic systems are given to demonstrate the effectiveness andfeasibility of the proposed complex network control and synchronization schemes.

Keywords: *Complex networks, Synchronization, Multiple time-varying delays, LMI, Adaptive control, Unified chaotic systems.*

1 Introduction

In the last few years, the research problem of complex networks has received a compelling attention from various different fields such as physics sciences, mathematics sciences, economic science and engineering application [1, 2], which extensive exist everywhere in our daily lives including power grids, ecosystems, food webs, and so on. Therefore, discussing the dynamical behaviour of complex networks is quite importance to comprehending the effects of the real-world network.

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A large amount of nonlinear dynamical behaviors of complex systems, synchronization is along-standing ever hot research topic [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. The large-scale system decomposition method was employed to discuss consensus complex network model [4]. Global asymptotic synchronization condition of neural networks with node delay and hybrid coupling was proposed by the Lyapunov functional method and LMI [6, 7, 8, 9, 10]. The complex dynamical network with neutral delay weights and adaptive coupling weights was also investigated [11, 13], respectively.

Nevertheless, according to above-mentioned relatively literatures, on the one hand, the coupling configuration weight matrix is constantly supposed to be symmetric and irreducible, which are very restrictive in engineering practice. On the other hand, it should be emphasized that a large of previous results on complex network synchronization were analysis the internal synchronization behavior of complex networks, which is called as “inner synchronization”. Instead of the synchronization between two or more complex networks is regarded as outer synchronization, which was firstly considered by Li et al. [16]. The complete outer synchronization (COS) problem of complex networks with identical topological structures was investigated by employing an open-plus-closed-loop controller (OPCL) scheme [16]. Other researchers also have investigated outer synchronization [17, 18, 19, 20, 21, 22, 23, 24, 25], etc. Afterwards, the outer synchronization for discrete-time complex networks was also investigated [17]. The complete synchronization of complex networks with nonidentical and identical topological structures were investigated by using adaptive control scheme [18, 19, 20], respectively.

There exist a large research results focusing on outer synchronization of complex networks [26, 27, 28, 29, 30, 31]. Soon after, the adaptive synchronization controller was proposed to achieve outer synchronization between complex networks with different strengths and nonidentical topological structures [31]. On the other hand, the outer synchronization also furnishes a platform and tool for identifying unknown network parameters or dynamical topology of a complex network [32, 33, 34].

In these early research results on network outer synchronization [26, 27, 28, 29, 30, 31, 32, 33, 34], it is often supposed that the each node in networks is identical, and the corresponding nodes in networks have completely the same dynamics. So, strictly speaking, it is COS between two complex networks. To overcome such a limitation, inverse outer synchronization (IOS) of complex networks was presented by constructing a nonlinear type OPCL [35]. The generalized outer synchronization (GOS) scheme of complex networks was also proposed [36]. A nonlinear synchronization controller and a adaptive updated law were presented to achieve the GOS between two complex networks with coupling time-varying delay [37].

Furthermore, for large-scale complex networks, since the finite information transmission and computer processing speeds among complex network nodes, time-delays are compelling attention and should be considered. Time-delay coupling extremely exists in many real-world systems such as gene regulatory networks and electrical power grids, etc. Therefore, it is quite important to investigate the synchronization of complex networks with coupling time-delay [38].

The problem of function projective synchronization for general complex networks with time delay was investigated [39]. Adaptive synchronization and pinning control of colored networks were considered [40]. Finite-time mixed outer synchronization controllers for complex networks with time-varying coupling delay were presented [41]. Finite-time stochastic outer synchronization between two complex networks with different topologies was considered again [42]. The outer synchronization between two complex networks with discontinuous coupling was investigated [43]. The inner synchronization was generalized by using input passivity and output passivity [44].

However, the investigation of synchronization of complex networks is still insufficient. On the one hand, the time-delay was always single in many these existing literatures [39, 41]. On the other hand, the multiple coupling time-varying delays were usually considered in inner synchronization [44]. To the best of the author's knowledge, there has been few results on outer synchronization of complex networks with multiple coupling time-varying delays until now, so how to solve the outer synchronization problem for complex networks with multiple coupling time-varying delays still remains largely and challenging.

Based on the foregoing discussions, in this paper, we propose theoretical analysis and numerical simulations for outer synchronization of complex networks with multiple coupling time-varying delays. The most important contribution of this paper is to establish some LMI synchronization conditions and to present a effective adaptive synchronization result for a general complex networks with multiple coupling time-varying delays.

The remaining sections of this paper are organized as follows. In Section 2, we formulate and describe the outer synchronization problem of complex networks with multiple coupling time-varying delays. In Section 3, some linear state feedback controllers are designed and the gain matrix is calculated analytically based on Lyapunov function stability and LMI. Adaptive synchronization scheme is achieved by adaptive control technology. Numerical simulations are given in Section 4. Finally, conclusions are presented in Section 5.

2 Problem description 2.1

Complex networks model

We consider a complex network consisting of N identical dynamical nodes with multiple coupling time-varying delays

$$\dot{x}_i(t) = f(x_i(t)) + \varepsilon \sum_{j=1}^{N} c_{ij}\Gamma x_j(t - \tau_j(t)), i = 1, 2, \cdots, N. \quad (1)$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t)]^T \in R^n$ is the state variables of the node i, $f(x_i(t)) = [f_1(x_i(t)), f_2(x_i(t)), \cdots, f_n(x_i(t))]^T \in R^n$ is a continuously differentiable nonlinear vector-valued function, $\varepsilon > 0$ is the coupling strength, $C = [c_{ij}] \in R^{n\times n}$ is the coupling configuration weight matrix representing the topological structure of the complex network, whose entries c_{ij} are defined as follows: if there is a connection from node j to node i $(j \neq i)$, $c_{ij} \neq 0$; otherwise, $c_{ij} = 0$ $(j \neq i)$, Γ is an inner-coupling matrix, $\tau_j(t) > 0$ is the coupling time-varying delay.

To achieve outer synchronization between two complex networks with multiple coupling time-varying delays, the complex network (1) is referred as the drive network, and the response network can be defined as follows

$$\dot{y}_i(t) = f(y_i(t)) + \varepsilon \sum_{j=1}^{N} c_{ij}\Gamma y_j(t - \tau_j(t)) + u_i(t), i = 1, 2, \cdots, N. \quad (2)$$

where $y_i(t) = [y_{i1}(t), y_{i2}(t), \cdots, y_{in}(t)]^T \in R^n$ is the state variables of the node i, $u_i(t) \in R^n$ is the linear synchronization controller for node i to be designed later.

Then, the outer synchronization error signal can be defined as follows

$$e_i(t) = y_i(t) - x_i(t). \quad (3)$$

Consequently, the error dynamical network between the drive network (1) and the response network (2) can be given by

$$\dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + \varepsilon \sum_{j=1}^{N} c_{ij}\Gamma e_j(t - \tau_j(t)) + u_i(t), \; i = 1, 2, \cdots, N. \tag{4}$$

2.2 Mathematical preliminaries

Before stating the main results of this paper, the following lemma and assumptions are necessary to prove our main results.

Assumption 1 *The nonlinear function $f(\cdot)$ satisfies the global Lipschitz condition; that is, there exists a constant $L > 0$ such that*

$$\| f(\alpha) - f(\beta)\| \le L \|\alpha - \beta\|, \forall \alpha, \beta \in R^n.$$

Remark 1 *It is easy to verify that some general chaotic systems such as the original Chuas circuit [45], hyperchaotic unified systems [46], unified chaotic systems [47] and modified Chua's circuit [45] can hold.*

Assumption 2 *The coupling time-varying delay $\tau_j(t)$ is a continuously differential function satisfies*

$$0 \le \dot{\tau}_j(t) \le \mu_j < 1, 0 \le \tau_j(t) \le \bar{\tau}_j.$$

Lemma 1 *(Matrix Cauchy inequality [48]) For any symmetric positive definite matrix $M \in R^{n\times n}$ and vector $x, y \in R^n$, yield*

$$\pm 2x^T y \le x^T Mx + y^T M^{-1} y.$$

Then, the complete outer synchronization between two complex networks can be defined as follows:

Definition 1 *The drive network (1) and the response network (2) are said to be achieved complete outer synchronization if there exists a controller $u_i(t)$ for any initial states $x_i(0), y_i(0)$ such that*

$$\lim_{t\to\infty} \| y_i(t, y_i(0)) - x_i(t, x_i(0)) \| = 0. \tag{5}$$

Remark 2 *From definition 1, we can see that outer synchronization means that the corresponding nodes of coupled networks could achieve complete synchronization regardless of synchronization behaviour of inner networks. In this paper, we go beyond any restrictions on the Laplacian matrix. In general, the Laplacian matrix is not assumed to be symmetric or irreducible. Therefore, these two complex networks can be undirected or directed and may also contain clusters and isolated nodes.*

2.3 Designing of controllers

First of all, we can design a nominal synchronization controller as follows

$$u_i(t) = Ke_i(t), i = 1, 2, \cdots, N. \tag{6}$$

where $K \in R^{n\times n}$ is the controller gain matrix to be designed later.

Second, we can design a non-fragile robust synchronization controller as follows

$$u_i(t) = (K+\Delta K)e_i(t), i = 1,2,\cdots,N. \tag{7}$$

where $\Delta K \in R^{n\times n}$ represents additive gain perturbation as follows:

$$\Delta K = M\Phi(t)N.$$

where M and N are known constant matrices, and the uncertain matrix $\Phi(t)$ satisfying the following condition

$$\Phi^T(t)\Phi(t) < I_n.$$

Thirdly, we can design a decoupling synchronization controller as follows

$$u_i(t) = ke_i(t), i = 1,2,\cdots,N. \tag{8}$$

where k is a constant gain to be designed later.

Finally, we can design a adaptive synchronization controller of the following form:

$$u_i(t) = -k_i(t)e_i(t), \tag{9}$$

$$\dot{k}_i(t) = y_i e_i^T(t)e_i(t), \quad i = 1,2,\cdots, N. \tag{10}$$

where $y_i > 0$ is the adaptive gain to be designed later.

Then, substitution controllers (6)-(9) into the error dynamics network (4), we can obtain controlled closed-loop error dynamical networks of the following forms

$$\dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + s\sum_{j=1}^{N} c_{ij}\Gamma e_j(t-\tau_j(t)) + Ke_i(t), \tag{11}$$

$$\dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + s\sum_{j=1}^{N} c_{ij}\Gamma e_j(t-\tau_j(t)) + (K+\Delta K)e_i(t), \tag{12}$$

$$\dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + s\sum_{j=1}^{N} c_{ij}\Gamma e_j(t-\tau_j(t)) + ke_i(t), \tag{13}$$

and

$$\dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + s\sum_{j=1}^{N} c_{ij}\Gamma e_j(t-\tau_j(t)) + k_i(t)e_i(t). \tag{14}$$

respectively.

3 Main Results

In this section, we are in the next position to propose our main results for outer synchronization of complex networks with multiple coupling time-varying delays constraint upon LMI and by using the adaptive control schemes.

3.1 Complete outer synchronization of complex networks

In this subsection, we first introduce a complete outer synchronization control strategy via LMI to present our main results.

Theorem 1 *Under assumptions 1-2, the drive network (1) and the response network (2) can achieve complete outer synchronization under the synchronization controller (6) if there exists a controller gain matrix* **K** *such that the following LMI condition holds*

$$I_N \otimes [K^T + K + 2LI_n] + diag(\frac{\varepsilon N}{1-\mu_1}, \frac{\varepsilon N}{1-\mu_2}, \cdots, \frac{\varepsilon N}{1-\mu_N}) \otimes I_n + \varepsilon^2([c^2_{ij}] \otimes \wp\wp^T) < 0. \quad (15)$$

Proof: Choose a Lyapunov-Krasovskii functional candidate for the error dynamical network (11) as follows

$$V(e(t)) = \sum_{i=1}^{N}[e^T_i(t)e_i(t) + \varepsilon N \int^t e^T_i(s)e_i(s)ds]. \quad (16)$$

Differentiating the Lyapunov-Krasovskii functional (16) along the trajectory of (11), yield

$$\dot{V}(e(t))|_{(11)} = \sum_{i=1}^{N}[2e^T_i(t)\dot{e}_i(t) + \frac{\varepsilon N}{1-\mu_i} e^T_i(t)e_i(t) - \frac{\varepsilon N(1-\dot{\tau}_i(t))}{1-\mu_i} e^T_i(t-\tau_i(t))e(t-\tau_i(t))] \quad (17)$$

$$= \sum_{i=1}^{N}[2e^T_i(t)[f(y_i(t)) - f(x_i(t))] + 2\varrho\sum_{j=1}^{N} c_{ij}e^T_i(t)\wp e(t-\tau_j(t)) + 2e^T_i(t)Ke_i(t)] + \frac{\varepsilon N}{1-\mu_i} e^T_i(t)e_i(t) - \frac{\varepsilon N(1-\dot{\tau}_i(t))}{1-\mu_i} e^T_i(t-\tau_i(t))e_i(t-\tau_i(t)).$$

Applying assumption 1, yield

$$e_i(t)^T[f(y_i(t))-f(x_i(t))] \le \| e_i(t) \| \cdot \| f(y_i(t))-f(x_i(t)) \| \le \| e_i(t) \| \cdot L \cdot \| e_i(t) \| = Le^T_i(t)e_i(t).$$

Applying lemma 1, yield

$$2\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}e^T_i(t)\wp e(t-\tau_j(t)) \le \sum_{i=1}^{N}\sum_{j=1}^{N} c^2_{ij}e^T_i(t)\wp\wp^T e_i(t) + \sum_{i=1}^{N}\sum_{j=1}^{N} e^T_j(t-\tau_j(t))e_j(t-\tau_j(t))$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} c^2_{ij}e^T_i(t)\wp\wp^T e_i(t) + N\sum_{i=1}^{N} e^T_i(t-\tau_i(t))e_i(t-\tau_i(t)).$$

Applying assumption 2, yield

$$1 - \frac{1-\dot{\tau}_i(t)}{1-\mu_i} < 0. \quad (20)$$

Substituting (18), (19) and (20) into (17), yield

$$\dot{V}(e(t))|_{(11)} \tag{21}$$

$$\leq \sum_{i=1}^{N} \{e_i^T(t)[2LI_n + K + K^T]e_i(t) + \varepsilon \sum_{j=1}^{N} c_{ij}^2 e_i^T(t) \wp \wp^T e_i(t) + \frac{\varepsilon N}{1-\mu_i} e_i^T(t) e_i(t)\}$$

$$\sum_{i=1}^{N} \{e_i^T(t)[2LI_n + K + K^T]e_i(t) + \frac{\varepsilon N}{1-\mu_i} e_i^T(t)e_i(t)\} + e^T(t)(\varepsilon[c_{ij}^2] \otimes \wp \wp^T)e(t)$$

$$= e^T(t)\{I_N \otimes [2LI_n + K + K^T] + diag(\frac{\varepsilon N}{1-\mu_1}, \frac{\varepsilon N}{1-\mu_2}, \cdots, \frac{\varepsilon N}{1-\mu_N}) \otimes I_n$$

$$+\varepsilon([c_{ij}^2] \otimes \wp \wp^T)\}e(t).$$

where $e(t) = [e_1^T(t), e_2^T(t), \cdots, e_N^T(t)]^T$.

According to (15)

$$\dot{V}(e(t)) < 0. \tag{22}$$

Thus, according to definition 1, the drive network (1) and the response network (2) have achieved complete outer synchronization under the controller (6).

The proof is completed.

Remark 3 *In a large of existing literature,* C *is supposed to be symmetric [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], which is not practical.* C *is supposed to be irreducible and reducible which are two parts to discuss [49], respectively. However, in this paper, no assumption on* C *is needed. The configuration matrix* C *needs not be symmetric, diffusive, or irreducible. This means that the networks (1) or (2) can either be undirected or directed networks and contain isolated nodes. The complex network structure in this paper is very general and this theorem can be applied to a great many complex networks in the real world.*

Theorem 2 *Under assumptions 1-2, the drive network (1) and the response network (2) can achieve complete outer synchronization under the nonfragile robust synchronization controller (7) if there exists a controller gain matrix* K *such that the following LMI condition holds*

$$\frac{}{1-\mu_1}, \frac{}{1-\mu_2}, \cdots, \frac{}{1-\mu_N}) \otimes I_n + \varepsilon^2([c_{ij}^2] \otimes \wp \wp^T) < 0.$$

Proof: The proof process is omitted since the conclusion is obvious.

Remark 4 *If there is no constraints of* (M, N)*, the non-fragile outer synchronization of complex networks reduces to theorem 1.*

$$I_N \otimes [K^T + K + 2LI_n + MM^T + N^T N] \tag{23}$$

$$+diag(\varepsilon N \quad \varepsilon N$$

Theorem 3 *Under assumptions 1-2, the drive network (1) and the response network (2) can achieve complete outer synchronization under the decoupling synchronization controller (8) if there exists a constant* k *such that the following condition holds*

$$k < -L - \frac{1}{2} \max_{1 \leq i \leq N} (\frac{\varepsilon N}{1-}) - \lambda_{max}(\frac{1}{2}, \varepsilon([4] \otimes \wp \wp^T)). \tag{24}$$

Proof: The proof process is omitted since the conclusion is obvious.

Remark 5 *Theorems 1-3 provide the outer synchronization condition of complex networks with multiple coupling time-varying delays. Similarly, outer exponential synchronization conditions can also be presented, which will be investigated in our future work.*

Remark 6 *Unlike from previous results on nonlinear feedback schemes for outer synchronization [26, 27, 28, 29, 30, 31, 32, 33, 34] which are lessflexible, we have designed a linear synchronization controller for achieving out synchronization. Our designed synchronization controller is not only robust, but also easy to implement.*

3.2 Adaptive outer synchronization of complex networks

In this subsection, we introducing a adaptive control strategy to present our main results:

Theorem 4 *Under assumptions 1-2, the drive network (1) and the response network (2) can achieve adaptive outer synchronization under the adaptive synchronization controller (9).*

Proof: Define a Lyapunov-Krasovskii functional candidate for the error dynamical network (13) as follows

$$V(e(t)) = \sum_{i=1}^{N}\Big[e_i^T(t)e_i(t) + \frac{1}{\gamma_i}(k_i(t)-\rho_i)^2 + \frac{\varepsilon N}{1-\mu_i}\int_{t-\tau_i(t)}^{t} e_i^T(s)e_i(s)ds\Big]. \tag{25}$$

Differentiating of Lyapunov-Krasovskii functional (25) along the trajectories (13), yield

$$\begin{aligned}
&\dot V(e(t))\big|_{(13)} \\
&= \sum_{i=1}^{N}\Big[2e_i^T(t)\dot e_i(t) + \frac{2}{\gamma_i}(k_i(t)-\rho_i)\dot k_i(t) + \frac{\varepsilon N}{1-\mu_i}e_i^T(t)e_i(t) - \frac{\varepsilon N(1-\dot\tau_i(t))}{1-\mu_i}e_i^T(t-\tau_i(t))e_i(t-\tau_i(t))\Big] \\
&= \sum_{i=1}^{N}\Big[2e_i^T(t)[f(y_i(t))-f(x_i(t))] + 2\varepsilon\sum_{j=1}^{N}c_{ij}e_i^T(t)\Gamma e_j(t-\tau_j(t))\Big] \\
&\quad -2k_ie_i^T(t)e_i(t) + \frac{2}{\gamma_i}(k_i(t)-\rho_i)\gamma_ie_i^T(t)e_i(t) \\
&\quad + \frac{\varepsilon N}{1-\mu_i}e_i^T(t)e_i(t) - \frac{\varepsilon N(1-\dot\tau_i(t))}{1-\mu_i}e_i^T(t-\tau_i(t))e_i(t-\tau_i(t)).
\end{aligned} \tag{26}$$

Then, it follows from (26), (18), (19) and (20) that

$$\begin{aligned}
&\dot V(e(t))\big|_{(13)} \\
&\le \sum_{i=1}^{N}\Big\{e_i^T(t)[2L-2\rho_i]e_i(t) + \varepsilon\sum_{j=1}^{N}c_{ij}^2e_i^T(t)\Gamma\Gamma^Te_i(t) + \frac{\varepsilon N}{1-\mu_i}e_i^T(t)e_i(t)\Big\} \\
&= e^T(t)\Big\{I_N\otimes[2L-2\rho_i] + \mathrm{diag}\Big(\frac{\varepsilon N}{1-\mu_1}, \frac{\varepsilon N}{1-\mu_2}, \cdots, \frac{\varepsilon N}{1-\mu_N}\Big)\otimes I_n \\
&\quad + \varepsilon([c_{ij}^2]\otimes\Gamma\Gamma^T)\Big\}e(t) \\
&= e(t)^T\Xi e(t).
\end{aligned} \tag{27}$$

where $\Xi = I_N \otimes [2L - 2\rho_i] + diag(\frac{\varepsilon N}{1-\mu_1}, \frac{\varepsilon N}{1-\mu_2}, \cdots, \frac{\varepsilon N}{1-\mu_N}) \otimes I_n + \varepsilon([c_{ij}^2] \otimes \Gamma\Gamma^T)$.

We can choose a sufficiently large ρ_i such that $\Xi < 0$, according to this

$$\dot V(e(t)) < 0. \tag{28}$$

Thus, according to Definition 1, the drive network (1) and the response network (2) have achieved adaptive synchronization under the adaptive synchronization controller (9)-(10).

The proof is completed.

Remark 7 *Some stability criteria for the synchronization between drive and response complex networks with multiple coupling time-varying delays are derived, which can also be applied to the complex network with single time delay. Thus, the results presented in this paper improve and generalize the corresponding results of recent works.*

4 Numerical simulations

In this section, numerical simulations are presented to verify the effectiveness and feasibility of the outer synchronization controller obtained in the previous section. Without loss of generality, we take unified chaotic systems as the local node dynamics.

4.1 Unified chaotic systems

Based on the Lorenz system, Lü system and Chen system, the unified chaotic system was proposed by Lü [47]. The unified chaotic system is described by

$$\begin{cases} \dot x_1 = (25\alpha + 1)(x_2 - x_1), \\ \dot x_2 = (28 - 35\alpha)x_1 - (29\alpha - 1)x_2 - x_1x_3, \\ \dot x_3 = x_1x_2 - \frac{8+\alpha}{3}x_3. \end{cases} \tag{29}$$

where $[x_1, x_2, x_3]^T \in R^3$ is the state variables group of unified chaotic system and $\alpha \in [0, 1]$ is a system parameter.

Due to the chaotic system (29) is chaotic for arbitrarily $\alpha \in [0, 1]$ and the system (29) belongs to the generalized Lorenz chaotic system for $0 \le \alpha < 0.8$, the system (29) belongs to the Lü chaotic system at $\alpha = 0.8$ and the system (29) belongs to the generalized Chen chaotic system for $0.8 < \alpha \le 1$, so the chaotic system (29) is regarded as unified chaotic systems.

In this subsection, the main theorems are illustrated by the following numerical simulations. For some special parameter values, from (29) , the parameter α is selected by $\alpha = 0.8$ and $\alpha = 0$, respectively, the initial condition $x(0) = [1, 0, -1]^T$ is chosen. The chaotic trajectories of unified chaotic system are illustrated in **Figures** 1, 3, respectively. The time evolution of state variable is shown in **Figures** 2, 4, respectively. It is easy to know that the trajectories of the unified chaotic system are bounded.

4.2 Systems simulation

Clearly, assumption 1 is verified if the $L = 150$, assumption 2 is verified if the $\tau_1(t) = \frac{1}{2} - \frac{1}{2}e^{-t}$ and $\bar\tau_1 = \frac{1}{2}$, $\mu_1 = \frac{1}{2}$, $\tau_2(t) = \frac{2}{5} - \frac{2}{5}e^{-t}$ and $\bar\tau_2 = \frac{2}{5}$, $\mu_2 = \frac{2}{5}$, $\tau_3(t) = \frac{3}{10} - \frac{3}{10}e^{-t}$ and $\bar\tau_3 = \frac{3}{10}$, $\mu_3 = \frac{3}{10}$.

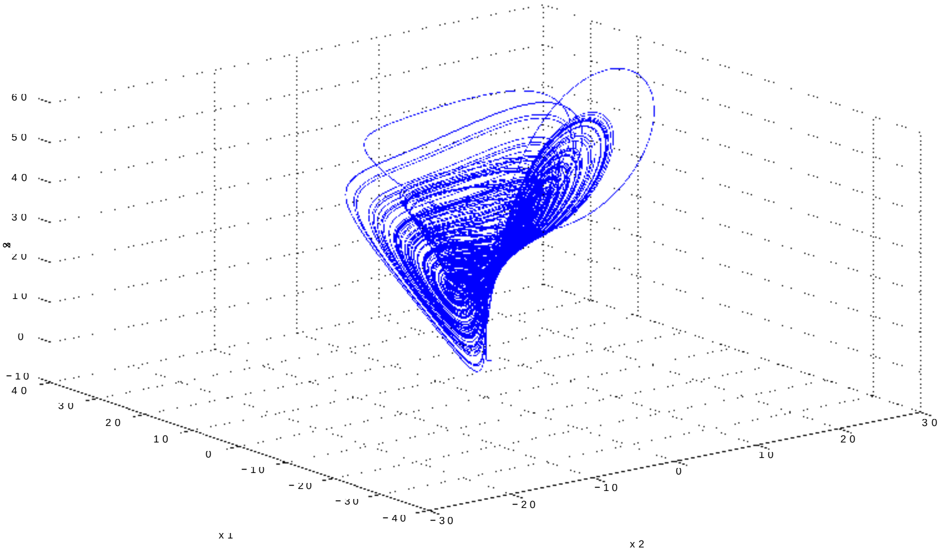

Figure 1. Chaotic attractor of unified chaotic systems (29) with α = 0.8**.**

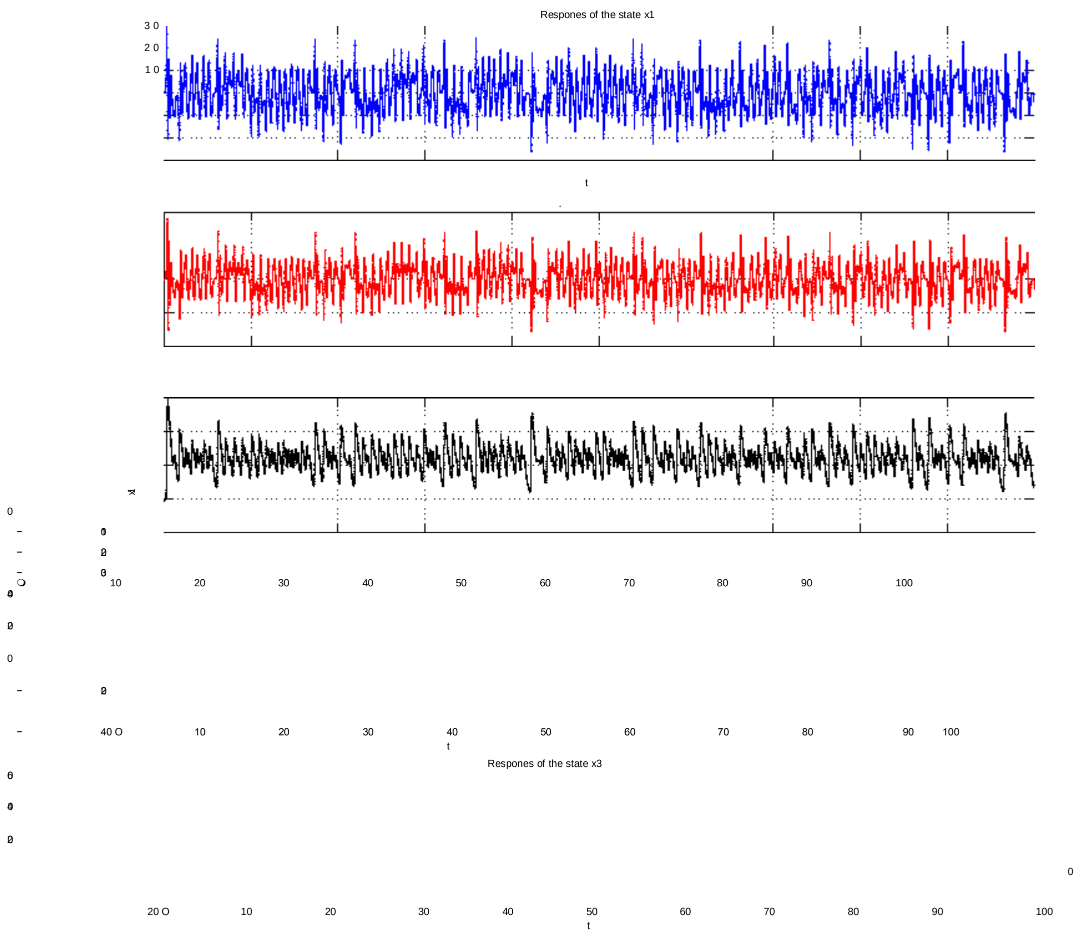

Figure 2. State response of unified chaotic systems (29) with $\alpha = 0.8$**.**

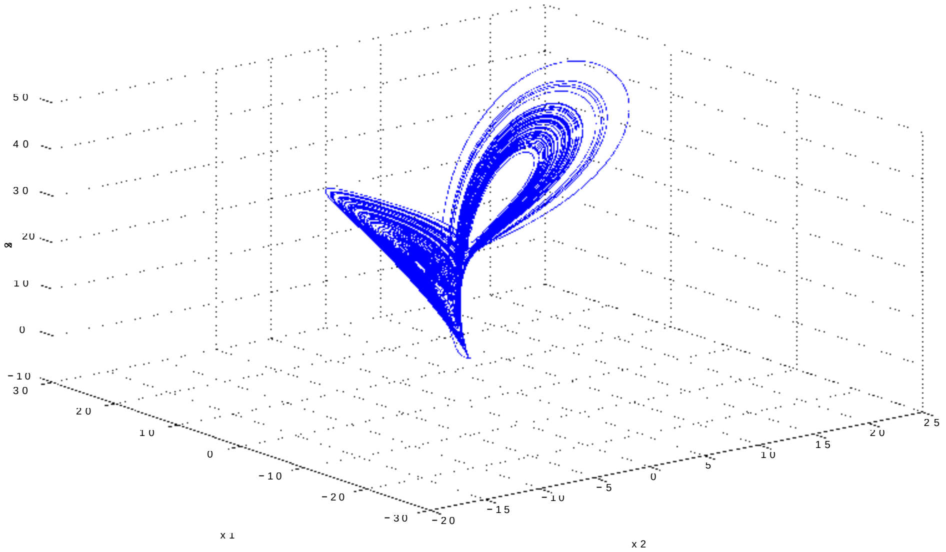

Figure 3. Chaotic attractor of unified chaotic systems (29) with α = 0.

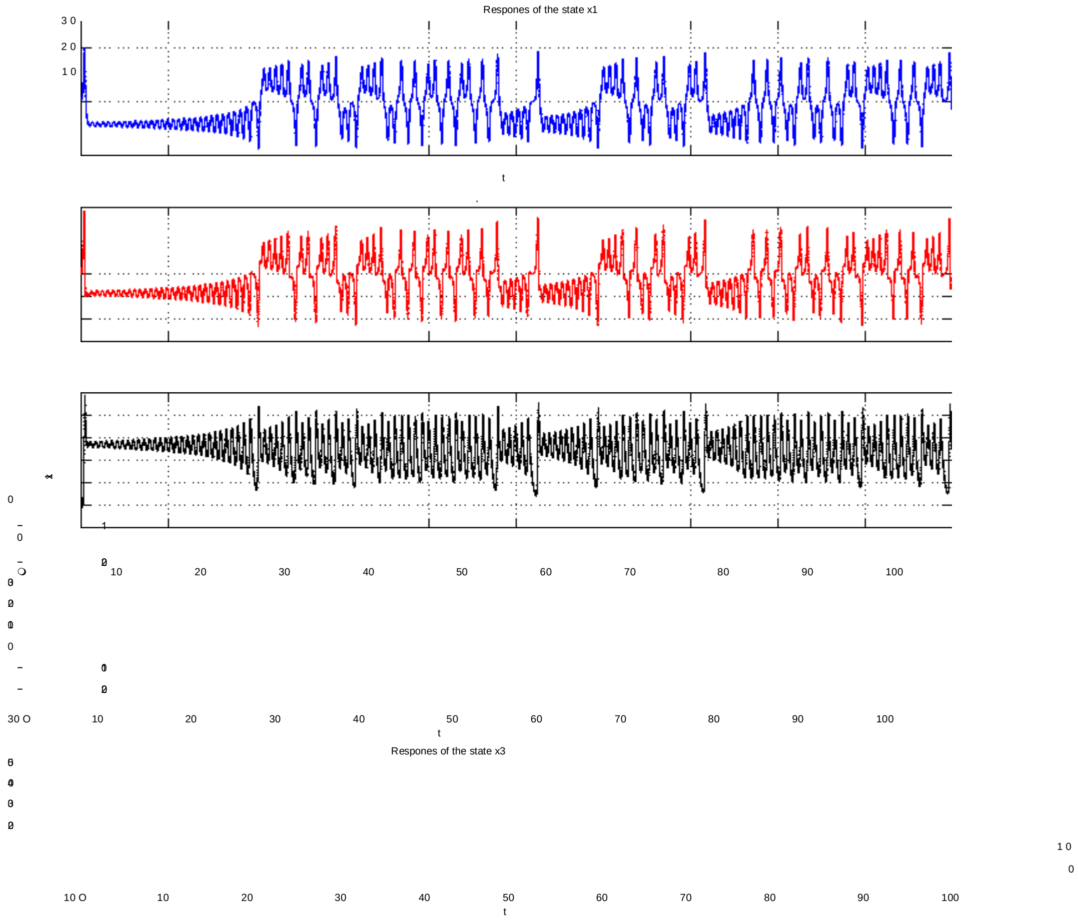

Figure 4. State response of unified chaotic systems (29) with $\alpha = 0$**.**

Example 1 *In this example, a diffusively coupled complex network containing 3 subsystems of (29) is investigated. The coupling configuration weight matrix* C *is chosen as follows*

$$C=\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

The inner coupling matrix Γ *and the coupling strength* ε *are chosen as follows*

$$\varepsilon = 1,$$
$$\Gamma = \begin{bmatrix} 1 & 4 & -1 \\ -2 & 3 & -4 \\ 2 & -1 & 5 \end{bmatrix}.$$

By using the Matlab LMI control Toolbox to solve the LMI condition in Theorem 1, we can obtain a set offeasible solutions,

$$K = \begin{bmatrix} -118.2570 & 0 & 0 \\ 0 & -118.8285 & 0 \\ 0 & 0 & -118.1885 \end{bmatrix}.$$

Therefore, we take the unified chaotic system ($\alpha = 0.8$) *as the local node dynamics, the drive network (1) and the response network (2) can achieve complete outer synchronization under the synchronization controller (6). With initial conditions*

$$x_1(0) = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, x_2(0) = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}, x_3(0) = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix},$$

$$y_1(0) = \begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix}, y_2(0) = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}, y_3(0) = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}.$$

The numerical simulations are proposed in ***Figures*** *5-7.*

From ***Figures*** *5-7, it is easy to know that, the synchronization errors can fast converge asymptotically to zero, which implies that the drive network (1) and the response network (2) have achieved complete outer synchronization.*

Example 2 *In this example, a non-diffusively coupled complex network containing 3 subsystems of (29) is investigated. The coupling configuration matrix* C *is chosen as follows*

$$C=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

The inner coupling matrix Γ *and the coupling strength* ε *are chosen as follows*

$$\varepsilon = 1,$$
$$\Gamma = \begin{bmatrix} 0 & 9 & 8 \\ 7 & 6 & 5 \\ 4 & 3 & 2 \end{bmatrix}.$$

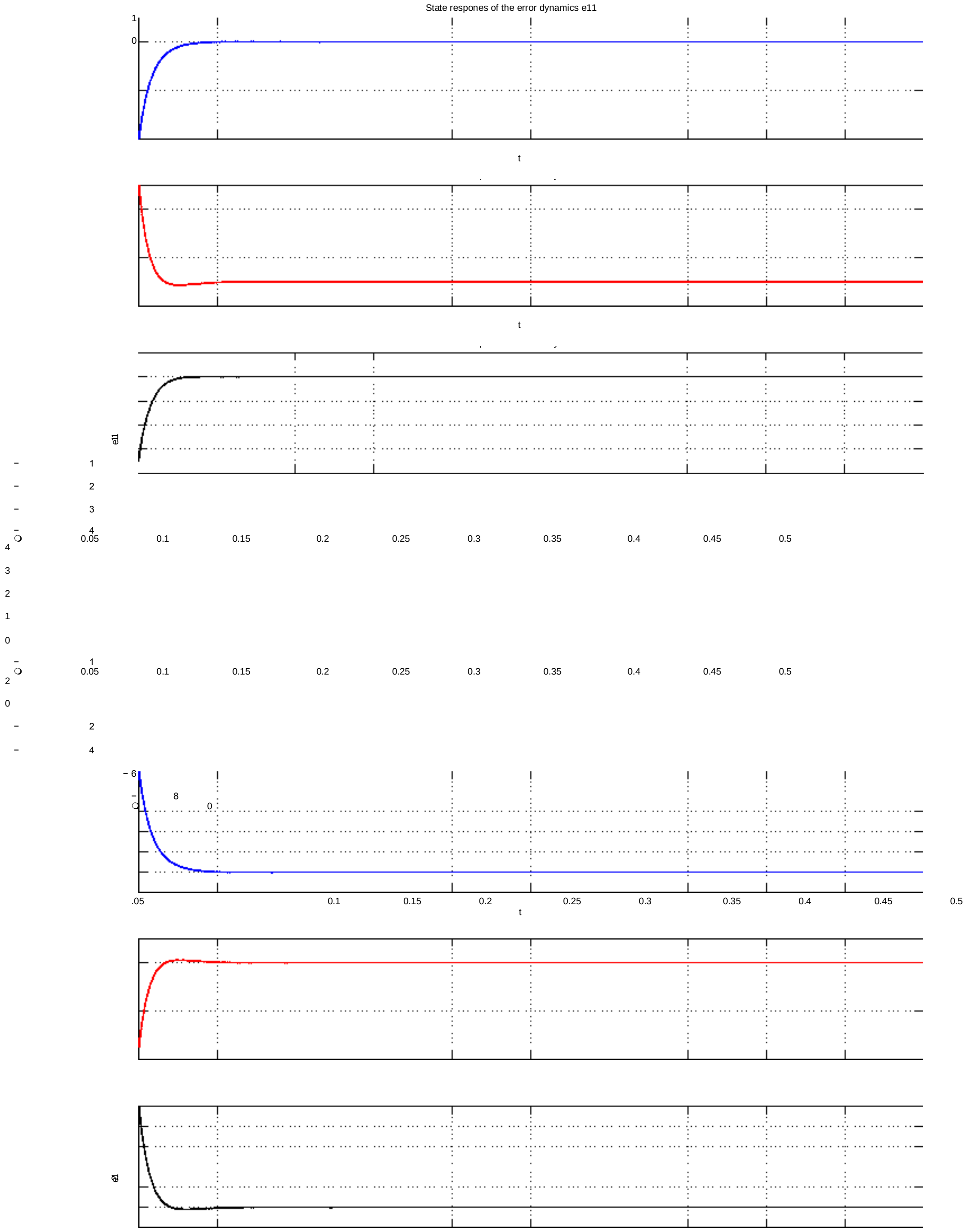

Figure 5. Complete outer synchronization errors $e_{12}(t)$ (i=1,2,3) of the drive network (1) and the response network (2).

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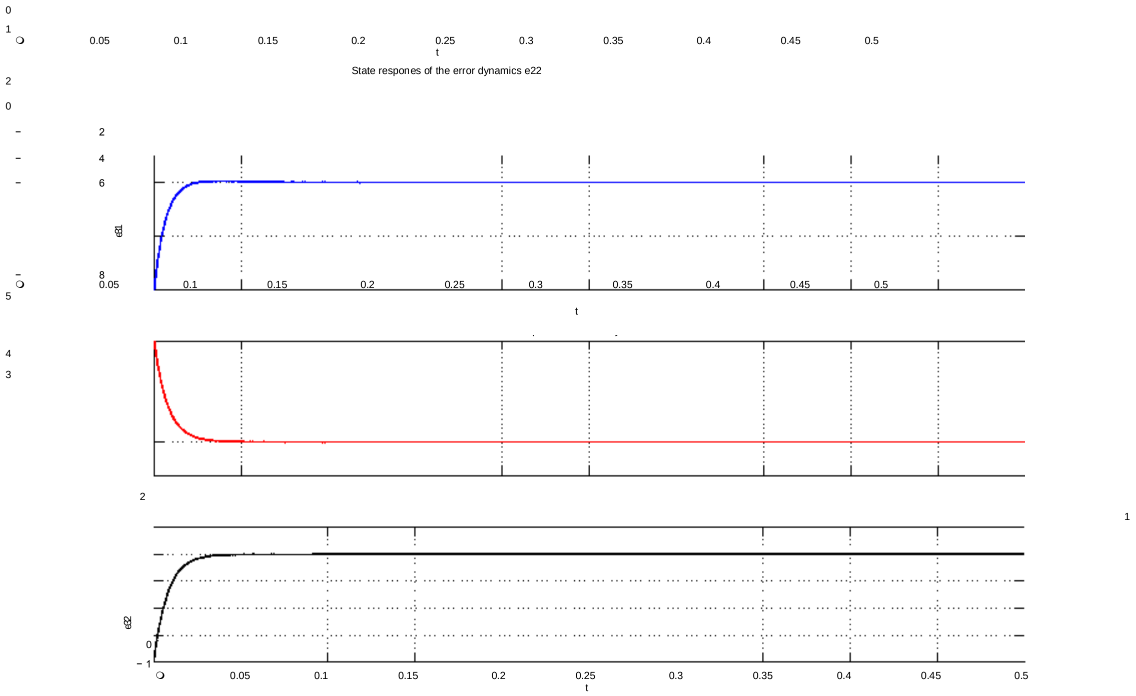


Figure 6. Complete outer synchronization errors $e_{22}(t)$ (i=1,2,3) of the drive network (1) and the response network (2).

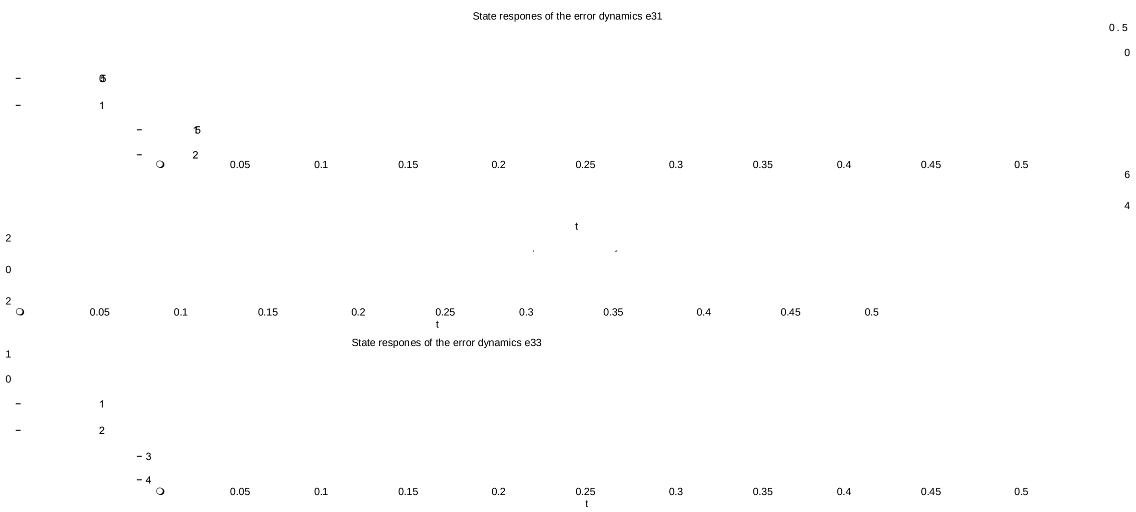
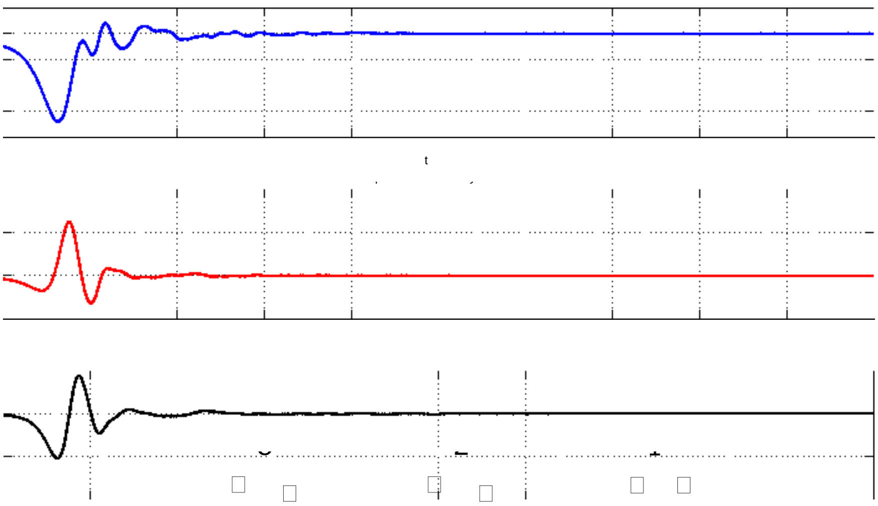


Figure 7. Complete outer synchronization errors $e_{3},(t)$ (i=1,2,3) of the drive network (1) and the response network (2).

Therefore, we take the unified chaotic system ($\alpha = 0$*) as the local node dynamics, the drive network (1) and the response network (2) can achieve adaptive outer synchronization* *under the adaptive synchronization controller (9)-(10). With initial conditions*



$x_1(0) = (6, 7, 3)^T, x_2(0) = (3, 9, 3)^T, x_3(0) = (3, 1, 8)^T$

$y_1(0) = (2, 5)^T, y_2(0) = (7, 1)^T, y_3(0) = (5, 3)^T$

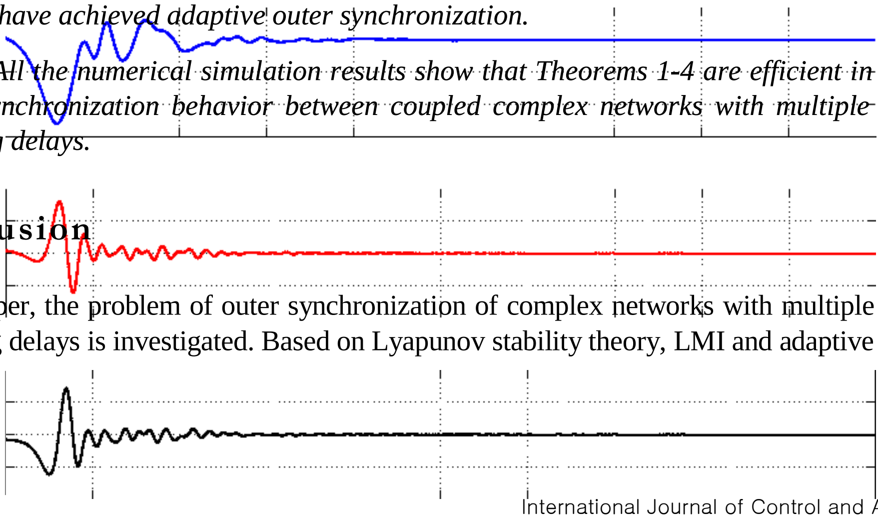
The numerical simulations are presented in ***Figures*** *8-10.*

From ***Figures*** *8-10, it is easy to know that the adaptive outer synchronization errors can fast converge asymptotically to zero, which implies that the drive network (1) and the response network (2) have achieved adaptive outer synchronization.*

Remark 8 *All the numerical simulation results show that Theorems 1-4 are efficient in ensuring the outer synchronization behavior between coupled complex networks with multiple coupling time-varying delays.*

5 Conclusion

In this paper, the problem of outer synchronization of complex networks with multiple coupling time-varying delays is investigated. Based on Lyapunov stability theory, LMI and adaptive control



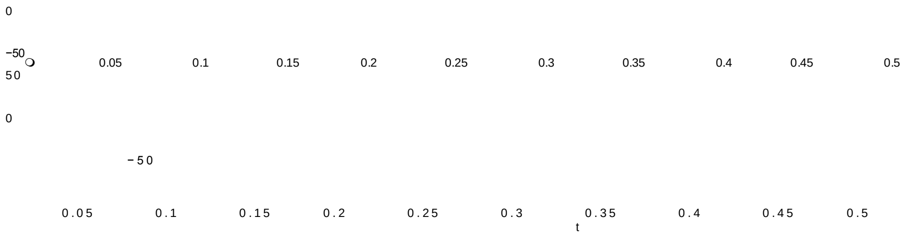


Figure 8. Adaptive outer synchronization errors $e_{12}(t)$ (i=1,2,3) of the drive network (1) and the response network (2).

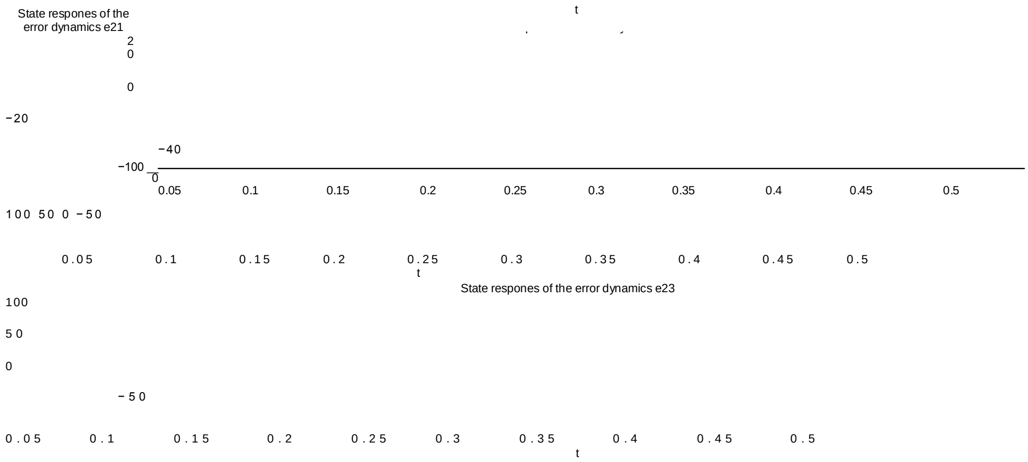
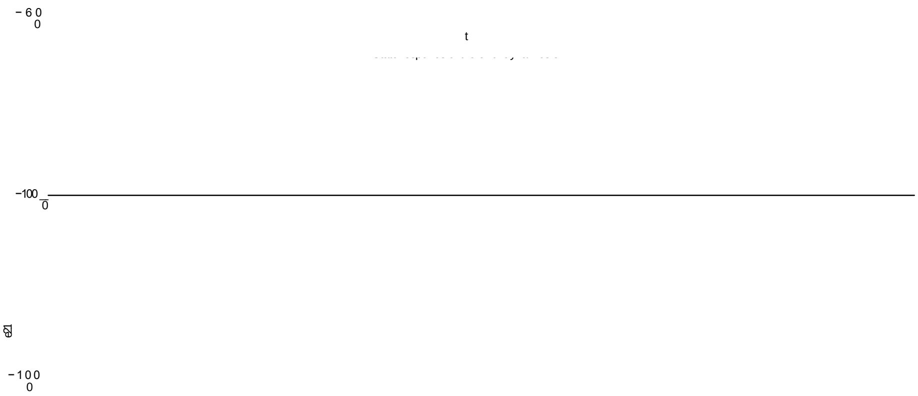


Figure 9. Adaptive outer synchronization errors $e_{2_2}(t)$ (i=1,2,3) of the drive network (1) and the response network (2).



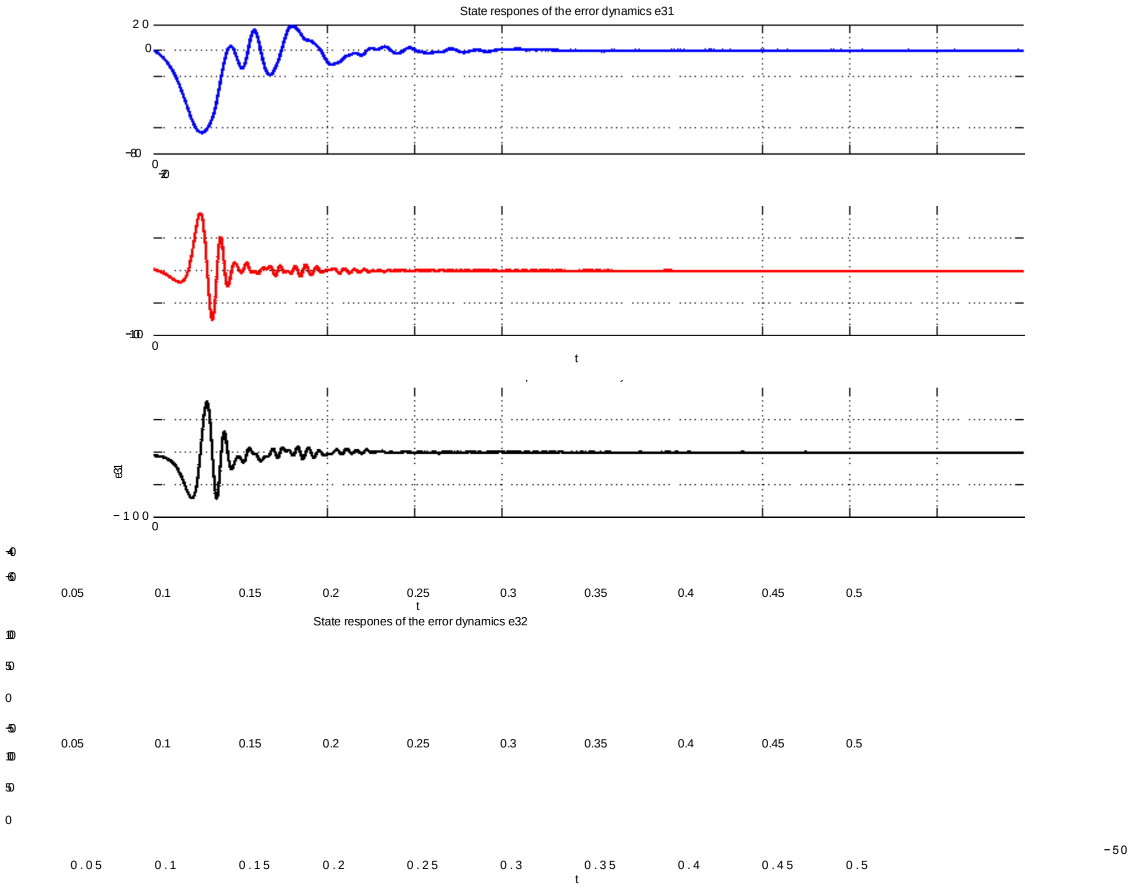


Figure 10. Adaptive outer synchronization errors $e_{3i}(t)$ (i=1,2,3) of the drive network (1) and the response network (2).

schemes, some novel synchronization criteria are derived, and some simple synchronization controllers are designed. The results presented in this paper improve and generalize the corresponding results of recent works. Numerical simulations are then presented to show the effectiveness and feasibility of the proposed complex networks control and synchronization schemes.

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